# Eisenstein Series over a Quadratic Imaginary Field

Martin Weissman

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**Abstract**

In this paper, I follow the notes of G. Shimura to discuss the Eisenstein series associated to the action of a modular group in $SL_2(\mathbf{C})$ on a 3-dimensional hyperbolic space.

## 1 Quaternions and Linear Fractional Transformations

Recall that the quaternions, denoted $\mathbf{H}$, are defined by $\mathbf{H} = \mathbf{R} \oplus \mathbf{R}i \oplus \mathbf{R}j \oplus \mathbf{R}k$, where $i$, $j$, $k$ satisfy the relations: $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$.

In addition to the usual quaternion conjugation, there are three important inner automorphisms of the quaternions. Namely, we have conjugation by the three elements $i$, $j$, $k$. Given a quaternion $z = a + bi + cj + dk$, we note that its conjugate is given by:

$$\bar{z} = a - bi - cj - dk,$$

and the inner automorphisms are given by:

$$izi^{-1} = a + bi - cj - dk,$$

$$jzj^{-1} = a - bi + cj - dk,$$

$$kzk^{-1} = a - bi - cj + dk.$$

There is a natural embedding: $\mathbf{R} \hookrightarrow \mathbf{C} = \mathbf{R} \oplus \mathbf{R}i \hookrightarrow \mathbf{H}$. This embedding will be implicitly understood throughout.

Now we consider the properties of $SL_2(\mathbf{C})$, naturally embedded in $GL_2(\mathbf{H})$. The following matrices will be used very often:

$$\epsilon = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \; j\epsilon = \epsilon j = \begin{bmatrix} 0 & -j \\ j & 0 \end{bmatrix}, \; k\epsilon = \epsilon k = \begin{bmatrix} 0 & -k \\ k & 0 \end{bmatrix}.$$

Also, note that for any $\alpha \in GL_2(\mathbf{H})$, ${}^t\alpha\epsilon\alpha = det(\alpha)\epsilon$. We now characterize $SL_2(\mathbf{C})$ alternately as a subset of $GL_2(\mathbf{H})$.

**Proposition 1** $SL_2(\mathbf{C}) = \{\alpha \in GL_2(\mathbf{H}) \mid {}^t\bar{\alpha}j\epsilon\alpha = j\epsilon, {}^t\bar{\alpha}k\epsilon\alpha = k\epsilon\}$.

PROOF: Let $H$ denote the set on the right hand side above, and suppose that $\alpha \in H$. Then we see that

$$j\epsilon\alpha\epsilon^{-1}j^{-1} = {}^t\bar{\alpha}^{-1} = k\epsilon\alpha\epsilon^{-1}k^{-1}.$$

Therefore $\epsilon\alpha\epsilon^{-1}$ commutes with $k^{-1}j = i$. From the properties of conjugation by $i$ above, this implies that $\epsilon\alpha\epsilon^{-1} \in GL_2(\mathbf{C})$. Hence $\alpha \in GL_2(\mathbf{C})$. Since the entries of $\alpha$ are complex, we see that ${}^t\bar{\alpha}j = j{}^t\alpha$, so that in particular,

$$j\epsilon = {}^t\bar{\alpha}j\epsilon\alpha = j{}^t\alpha\epsilon\alpha.$$

From this, wee see that ${}^t\alpha\epsilon\alpha = \epsilon$. Thus $\alpha \in SL_2(\mathbf{C})$. Hence $H \subseteq SL_2(\mathbf{C})$. For the converse, take $\alpha$ in $SL_2(\mathbf{C})$. Then by similar arguments, we have:

$${}^t\bar{\alpha}j\epsilon\alpha = j{}^t\alpha\epsilon\alpha = j\epsilon,$$

and likewise for $k$. Hence $\alpha \in H$. Thus $SL_2(\mathbf{C}) \subseteq H$ so $H = SL_2(\mathbf{C})$.

Q.E.D.

Let $G$ hereafter denote the group $SL_2(\mathbf{C})$, keeping in mind the above formulation. Let us now define the action of $G$ on a suitable upper half space in $\mathbf{H}$. This upper half space is defined by:

$$S = \{a + bi + cj \in \mathbf{H} | a, b, c \in \mathbf{R}, c > 0\}.$$

Let $\rho$ denote the automorphism of $\mathbf{H}$ given by $z^\rho = izi^{-1}$. Then $S^\rho$ is just the half-space opposite of $S$, whose elements have negative $j$ component rather than positive.

To understand the action of $G$ on $S$, we first study the following sets of matrices. Using them, the desired action and the factors of automorphy arise naturally from the next proposition.

$$X = \left\{ \xi \in M_2(\mathbf{H}) \mid {}^t\bar{\xi}j\epsilon\xi = \begin{bmatrix} a & 0 \\ 0 & -d \end{bmatrix} a, d > 0, {}^t\bar{\xi}k\epsilon\xi = \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix} \right\},$$

$$V = \left\{ \eta \in \mathbf{H}^2 \mid {}^t\bar{\eta}j\epsilon\eta < 0, {}^t\bar{\eta}k\epsilon\eta = 0 \right\}.$$

**Proposition 2** *The following maps are bijections:*

*(1)* $S \times \mathbf{H}^\times \to V$ *via* $(z, \mu) \mapsto \begin{bmatrix} z\mu \\ \mu \end{bmatrix}$;

*(2)* $S \times \mathbf{H}^\times \times \mathbf{H}^\times \to X$ *via* $(z, \lambda, \mu) \mapsto \begin{bmatrix} z^\rho & z \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$.

PROOF: First, suppose that $\eta = \begin{bmatrix} x \\ y \end{bmatrix} \in V$. Then we see from the definition of $V$ that:

(a) $\begin{bmatrix} \bar{x} & \bar{y} \end{bmatrix} \begin{bmatrix} 0 & -k \\ k & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \bar{y}kx - \bar{x}ky = 0,$

(b) $\bar{y}jx - \bar{x}jy < 0.$

From (b), we see that $xy \neq 0$. Let $z = xy^{-1}$. From (a), we see that $kzk^{-1} = \bar{z}$. Hence $z \in \mathbf{C} \oplus \mathbf{R}j$. Moreover from (b), we see that $jz - \bar{z}j = \bar{y}^{-1}(\bar{y}jx - \bar{x}jy)y^{-1} < 0$. Hence $z \in S$. Therefore $\eta = \begin{bmatrix} zy \\ y \end{bmatrix}$. Conversely, it is easy to show that given $z \in S$ and $y \in \mathbf{H}^\times$, the matrix $\begin{bmatrix} zy \\ y \end{bmatrix}$ is in $V$. This finishes the proof of (1). Next, suppose $\xi = \begin{bmatrix} u & x \\ v & y \end{bmatrix} \in X$. Then we may see that $\begin{bmatrix} x \\ y \end{bmatrix}$ is in $V$. We have from the definition of $X$ that

$$\begin{bmatrix} \bar{u} & \bar{v} \\ \bar{x} & \bar{y} \end{bmatrix} \begin{bmatrix} 0 & -j \\ j & 0 \end{bmatrix} \begin{bmatrix} u & x \\ v & y \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & -d \end{bmatrix}.$$

Hence $\begin{bmatrix} \bar{u} & \bar{v} \\ \bar{x} & \bar{y} \end{bmatrix} \begin{bmatrix} -jv & -jy \\ ju & jx \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & -d \end{bmatrix}$. Looking at the entries of this and the similar equation involving $k$ yields the following equations:

$$-\bar{u}jy + \bar{v}jx = 0,$$

$$-\bar{u}kv + \bar{v}ku = 0,$$

$$-\bar{u}jv + \bar{v}ju > 0.$$

From the last of these, we see that $uv \neq 0$. Now letting $z = xy^{-1}$, we arrive at $kuv^{-1}k^{-1} = \bar{v}^{-1}\bar{u} = jxy^{-1}j^{-1}$ so that $uv^{-1} = z^\rho$. Thus we have $\xi = \begin{bmatrix} z^\rho & z \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & y \end{bmatrix}$ The converse may be seen by elementary computation as well, so we have proven (2).

Q.E.D.

Now let us see the importance of $X$ and $V$ in considering linear fractional transformations. First note that $G$ acts on $X$ and $V$ by left multiplication. Thus given any $\alpha \in G$, and $z \in S$, by our previous proposition, we have, for some $w(\alpha, z) \in S$ and some $\lambda(\alpha, z), \mu(\alpha, z)$:

$$\alpha \begin{bmatrix} z^\rho & z \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} w^\rho & w \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda(\alpha, z) & 0 \\ 0 & \mu(\alpha, z) \end{bmatrix}. \qquad (1)$$

Then we see that the map $z \mapsto w(\alpha, z)$ gives a bona fide action of $G$ on $S$, so hereafter we denote $\alpha z = w(\alpha, z)$. We call the functions $\lambda$ and $\mu$ the factors of automorphy. Explicitly, given elements: $\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G, z \in S$, we may compute:

$$\alpha z = (az + b)(cz + d)^{-1},$$

$$\lambda(\alpha, z) = cz^\rho + d = (cz + d)^\rho,$$

$$\mu(\alpha, z) = cz + d.$$

Note also that $G$ acts also on $S^\rho$, and $\alpha(z^\rho) = (\alpha z)^\rho$. In other words, the sign of the $j$ component of an element of $S$ or $S^\rho$ is invariant under the action of $G$. We may look more closely at the effect of the action upon the $j$ component as follows:

Given $z \in S$, let $\eta(z)$ denote the $j$ component of $z$. Note that this is given explicitly by $\eta(z) = \bar{z}j - jz$. Then we see that if $\xi = \begin{bmatrix} z^\rho & z \\ 1 & 1 \end{bmatrix}$, then we have:

$$^t\bar{\xi}j\epsilon\xi = \begin{bmatrix} 2\eta(z) & 0 \\ 0 & -2\eta(z) \end{bmatrix}.$$

By Equation (1) this yields:

$$\left[\begin{array}{cc} \eta(z) & 0 \\ 0 & -\eta(z) \end{array}\right] = \left[\begin{array}{cc} \overline{\lambda(\alpha,z)} & 0 \\ 0 & \overline{\mu(\alpha,z)} \end{array}\right] \left[\begin{array}{cc} \eta(\alpha z) & 0 \\ 0 & -\eta(\alpha z) \end{array}\right] \left[\begin{array}{cc} \lambda(\alpha,z) & 0 \\ 0 & \mu(\alpha,z) \end{array}\right].$$

Hence we arrive at the explicit result:

$$\eta(\alpha(z)) = |\mu(\alpha,z)|^{-2}\eta(z) = |\lambda(\alpha,z)|^{-2}\eta(z). \tag{2}$$

As a by-product of this equation, note that $|\lambda(\alpha,z)| = |\mu(\alpha,z)|$.

Let us now use these properties of the action of $G$ on $S$ to derive some results on the structure of $S$ as it relates to the action. Namely, we now investigate the structure of $S$ as a quotient of $G$ by an appropriate subgroup, and the invariant measure and metric of $S$ with respect to $G$.

Let us define the parabolic subgroup of $G$, $P = \left\{ \left( \begin{array}{cc} a & b \\ 0 & a^{-1} \end{array} \right) \in G \right\}$. Then, we may describe the action of $G$ on $S$ with the following:

**Proposition 3** *(1)* $\{\alpha \in G | \alpha(j) = j\} = SU(2)$,

*(2) $G$ acts transitively on $S$,*

*(3)* $G = P \cdot SU(2)$,

*(4)* $S \cong SL_2(\mathbf{C})/SU(2)$.

PROOF: For (1), suppose that $\alpha = \left[\begin{array}{cc} a & b \\ c & d \end{array}\right] \in G$. Then we see that:

$$\begin{aligned} \alpha(j) = j \quad &\Longleftrightarrow \quad aj + b = j(cj+d) = -\bar{c} + \bar{d}j \\ &\Longleftrightarrow \quad \alpha = \left[\begin{array}{cc} a & b \\ -\bar{b} & \bar{a} \end{array}\right] \\ &\Longrightarrow \quad \alpha \in SU(2). \end{aligned}$$

For the last step, note that since $\alpha \in G$, we know that $|a|^2 + |b|^2 = 1$. Conversely, if $\alpha \in SU(2)$, then it is easy to show that $\alpha(j) = j$.

For (2), Suppose that $u + vj \in S$, with $u \in \mathbf{C}$, $v \in \mathbf{R}_+$. Then define:

$$\alpha = \left[\begin{array}{cc} 1 & u \\ 0 & 1 \end{array}\right] \left[\begin{array}{cc} v^{\frac{1}{2}} & 0 \\ 0 & v^{-\frac{1}{2}} \end{array}\right].$$

Then we know that $\alpha \in G$, and $\alpha j = u + vj$. By our last result, this is sufficient to show that $G$ is transitive on $S$.

For (3), first note that clearly $P \cdot SU(2) \subset G$. Furthermore, given $\alpha \in G$, such that $\alpha j = w$, there exists an element $\beta \in P$ such that $\beta j = w$, as in the proof of (2). Then we see that $\beta^{-1}\alpha j = j$, so that $\beta^{-1}\alpha = \gamma$ for some $\gamma \in SU(2)$ by (1). Hence $\alpha = \beta\gamma$, so $G = P \cdot SU(2)$.

Now (4) follows directly from (1), (2), and (3), using basic properties of topological groups.

Q.E.D.

Now we investigate the metric properties of $S$ induced by this group action.

**Lemma 1** $d(\alpha z) = k\overline{\mu(\alpha, z)}^{-1} k^{-1} dz \mu(\alpha, z)^{-1}$.

PROOF: By equation (1), we may deduce that:

$$\begin{bmatrix} * & j(z - z_1) \\ * & * \end{bmatrix} = \begin{bmatrix} \overline{\mu(\alpha, z_1)}^{\rho} & 0 \\ 0 & \overline{\mu(\alpha, z_1)} \end{bmatrix} \begin{bmatrix} * & j(\alpha z - \alpha z_1) \\ * & * \end{bmatrix} \begin{bmatrix} \mu(\alpha, z)^{\rho} & 0 \\ 0 & \mu(\alpha, z) \end{bmatrix}.$$

Multiplying the matrices on the right hand side and equating yields:

$$\begin{aligned} j(z - z_1) &= \overline{\mu(\alpha, z_1)}^{\rho} j(\alpha z - \alpha z_1)\mu(\alpha, z), \\ z - z_1 &= j^{-1} i \overline{\mu(\alpha, z_1)} i^{-1} j(\alpha z - \alpha z_1)\mu(\alpha, z). \end{aligned}$$

Letting $z_1$ tend to $z$ gives us the differential:

$$dz = k\overline{\mu(\alpha, z)} k^{-1} d(\alpha z)\mu(\alpha, z),$$

so that we may solve for $d(\alpha z)$ as:

$$d(\alpha z) = k\overline{\mu(\alpha, z)}^{-1} k^{-1} dz \mu(\alpha, z)^{-1}.$$

Q.E.D.

**Proposition 4** *Identifying an element $\alpha$ with its action upon $S$, we have:* $|jacob(\alpha)| = |\mu(\alpha, z)|^{-6}$.

PROOF: Consider the space $\tilde{S} = \mathbf{C} \oplus \mathbf{R}j \subset \mathbf{H}$. Let $\phi : \mathbf{H} \longrightarrow \mathbf{H}$ be given by $\phi(x) = k\bar{\lambda}k^{-1}x\lambda$ for some $\lambda \in \mathbf{H}^{\times}$. Then we may observe that $\tilde{S}$ is stable under $\phi$; for since $\tilde{S} = \{x \in \mathbf{H} \mid kxk^{-1} = \bar{x}\}$, and given $x \in \tilde{S}$,

$$k\phi(x)k^{-1} = k \cdot k\bar{\lambda}k^{-1}x\lambda k^{-1} = \bar{\lambda}\bar{x}k\lambda k^{-1} = \overline{k\bar{\lambda}k^{-1}x\lambda} = \overline{\phi(x)},$$

we have $\phi(x) \in \tilde{S}$. Thus we see that the jacobian $jacob(\phi) = |\lambda|^6$ on $\tilde{S}$. Putting $\lambda = \mu(\alpha, z)^{-1}$ and applying our last lemma yields the desired result.

Q.E.D.

The Haar measure on $S$ induced by this group action then is the unique measure $\mu$ satisfying

$$\int_S \chi_R(z)d\mu = \int_S \chi_R(\alpha(z))d\mu,$$

for all open sets $R$, where $\chi_R$ denotes the characteristic function of $R$. Then, our last proposition together with equation (2) gives us the following invariant measure, with $z = x + yi + vj$:

$$d\mu = \frac{|dx \wedge dy \wedge dv|}{\eta(z)^3}.$$

Along similar lines, we may also compute the invariant metric on $S$ to be $\eta(z)^{-2}(dx^2 + dy^2 + dv^2)$.

## 2 The K-Bessel Function

The important properties of Eisenstein series are derived from the analytic properties of the K-Bessel function. In this section, we study the properties of the K-Bessel function, and use them to compute an important Fourier transform.

Define the K-Bessel function for all $y > 0$, $s \in \mathbf{C}$ by:

$$K(s,y) = \int_0^{\infty} e^{-y(t+t^{-1})}t^{s-1}dt.$$

It will also be convenient to define a normalized version of the K-Bessel function, given by:

$$K_s(y) = \frac{1}{2}\int_0^{\infty} e^{-\frac{1}{2}y(t-t^{-1})}t^{s-1}dt.$$

It is related to our original K-Bessel function via $K(s,y) = 2K_s(2y)$, and more generally, we may see that for $a \in \mathbf{C}$,

$$K_s(ay) = \frac{1}{2}a^{-s}\int_0^\infty e^{-\frac{1}{2}y(t-a^2t^{-1})}t^{s-1}dt.$$

The K-Bessel functions will arise in our context as the unique solutions of a differential equation. Namely, $f(y) = K_s(y)$ is a solution of

$$y^2f''(y) + yf'(y) - (y^2+s^2)f = 0.$$

Also, the function $k(y) = yK_s(ay)$ satisfies

$$y^2k''(y) - yk'(y) - (a^2y^2+s^2-1)k = 0. \tag{3}$$

Under sufficient growth conditions, we will later see that this last solution is picked out uniquely. It will be important to know that for $s$ within a compact set, we have as $y$ gets large,

$$|K(s,y)| \leq C(e^{-2y}). \tag{4}$$

where $C$ depends on the set that we allow $s$ to vary on. To see this, note that $t + t^{-1} = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2 + 2$. Hence we see that:

$$\begin{aligned} |K(s,y)| &\leq e^{-2y}\int_0^\infty e^{-y(t^{\frac{1}{2}}-t^{-\frac{1}{2}})^2}|t^{s-1}|dt \\ &\leq e^{-2y}\int_0^\infty e^{-y_0(t^{\frac{1}{2}}-t^{-\frac{1}{2}})^2}|t^{s-1}|dt \\ &\leq e^{-2y}e^{2y_0}K(Re(s),y_0), \end{aligned}$$

where we assume $y \geq y_0$. Hence for $s$ within a compact set, we see that $|K(s,y)| \leq C(e^{-2y})$, for all $y \geq y_0$.

There are three special properties of the K-Bessel functions that we will need. The first one is:

$$K(s,y) = K(-s,y). \tag{5}$$

To see this, just make the substitution $t \mapsto t^{-1}$.

The second property is critical to the relationship between the K-Bessel function and the Eisenstein series through the Fourier transform. It expresses a certain integral in terms of the K-Bessel function. Namely, we have:

$$\int_0^\infty e^{-(a^2t+b^2t^{-1})}t^{s-1}dt = \left(\frac{b}{a}\right)^s K(s,ab). \tag{6}$$

This can be seen via the substitution $u = |\frac{a}{b}|\, t$, and note that we implicitly assumed $ab \neq 0$.

Our third property allows us to understand the arithmetic properties of the Fourier coefficients of Eisenstein series. It expresses the derivatives of the K-Bessel function with respect to $y$ in a form very well adapted to arithmetic computation.

Integration by parts on the definition of $K(s,y)$ yields:

$$\begin{aligned} K(s,y) &= \left[ e^{-y(t+t^{-1})} \frac{t^s}{s} \right]_0^\infty - \int_0^\infty e^{-y(t+t^{-1})}(-y)(1-t^{-2})t^{s-1}dt \\ &= \frac{y}{s}\int_0^\infty e^{-y(t+t^{-1})}(t^{s+1}-t^{s-1})\frac{dt}{t} \\ &= \frac{y}{s}[K(s+1,y) - K(s-1,y)]. \end{aligned}$$

Moreover, if we let $K^{(m)}(s,y) = \left(\frac{\partial}{\partial y}\right)^m K(s,y)$, then we may see that in particular:

$$\begin{aligned} K^{(1)}(s,y) &= \int_0^\infty e^{-y(t+t^{-1})}(-1)(t+t^{-1})t^{s-1}dt \\ &= -K(s+1,y) - K(s-1,y), \end{aligned}$$

or, in general, we have the formula which is our aforementioned third property:

$$K^{(m)}(s,y) = (-1)^m \sum_{k=0}^{m} \binom{m}{k} K(s+m-2k, y). \qquad (7)$$

We are now in a position to show that the function $k(y) = yK_s(ay)$ is picked out uniquely by it's differential equation (3), given appropriate growth conditions. Specifically we have:

**Proposition 5** *Suppose that $k(y)$ satisfies the differential equation $y^2k''(y) - yk'(y) - (a^2y^2 + s^2 - 1)k = 0$ as above, and also that $k(y) = O(y^\beta)$ for some $\beta \in \mathbf{R}$ as $y \to \infty$. Then $k(y) = Cf_a(y)$ where $f_a(y) = yK_s(ay)$ for some constant $C$.*

PROOF: It is not difficult to check that our function $f_a(y) = yK_s(ay)$ satisfies the differential equation, with the aid of equation (7). Moreover, it satisfies the growth condition, since $yK_s(ay) = \frac{y}{2}K(s, \frac{y}{2})$, and we may apply the growth estimate (4). Now given a function $k$ as above, put $h = f_ak' - f_a'k$. Then $h' = f_ak'' - f_a''k = y^{-1}h$ hence $h = cy$ for some constant $c$. Now, we know that $f$ and $k'$ are both $O(y^D)$ with $D \in \mathbf{R}$, and we may also estimate $f_a$ to be $O(e^{-|a|y/2})$, so that $f_a k' - f_a' k = ay$ implies $a = 0$. Hence $k$ is a constant multiple of $f_a$ as desired.

Q.E.D.

Now, let us use these three properties to perform an important calculation: the Fourier transform of $(x^2 + y^2 + v^2)^{-s}$ where $v$ is held constant.

Consider the following integral:

$$\int_{\mathbf{R}^2} \int_0^\infty e^{-\pi t(x^2+y^2+v^2)} e^{-2\pi i(xa+yb)} t^{s-1} dt.$$

We evaluate it in two ways, by switching the order of integration. Integrating with respect to $t$ first relates this integral to the desired Fourier transform. Integrating first with respect to $x$ and $y$ relates the integral to the K-Bessel function. First we integrate with respect to $t$:

$$\int_{\mathbf{R}^2} \int_0^\infty e^{-\pi t(x^2+y^2+v^2)} e^{-2\pi i(xa+yb)} t^{s-1} dt$$
$$= \int_{\mathbf{R}^2} \Gamma(s) \pi^{-s} (x^2 + y^2 + v^2)^{-s} e^{-2\pi i(xa+yb)} dx dy.$$

Now switching the order of integration, we may evaluate a standard Gaussian integral to get:

$$\int_0^\infty \int_{\mathbf{R}^2} e^{-\pi t(x^2+y^2+v^2)} e^{-2\pi i(xa+yb)} t^{s-1} dt \tag{8}$$
$$= \int_0^\infty t^{s-2} e^{-\pi(tv^2+t^{-1}(a^2+b^2))} dt \tag{9}$$
$$= \left(\frac{|a+bi|}{v}\right)^{s-1} K(s-1, \pi v|a+bi|). \tag{10}$$

To arrive at the last step, we assume that $a^2 + b^2 \neq 0$ and apply our third property, equation (7). In the case that $a = b = 0$, we may integrate equation (9) to get:

$$\int_0^\infty t^{s-2} e^{-\pi(tv^2+t^{-1}(a^2+b^2))} dt = \Gamma(s-1)\pi^{1-s} v^{2(1-s)}. \tag{11}$$

We may put together all of these results to arrive at the Fourier transform of $(x^2 + y^2 + v^2)^{-s}$. Let $F_s(a, b)$ denote this Fourier transform, so that explicitly, we have:

$$F_s(a, b) = \int_{\mathbf{R}^2} (x^2 + y^2 + v^2)^{-s} e^{-2\pi i(xa+yb)} dx dy.$$

Then from our previous results, we arrive at the following two cases:

(1) Suppose $a^2 + b^2 \neq 0$ and $Re(s) > 0$. Then we have:

$$F_s(a,b) = \frac{\pi^s}{\Gamma(s)} \left( \frac{|a+bi|}{v} \right)^{s-1} K(s-1, \pi v |a+bi|).$$

(2) Suppose $a = b = 0$ and $Re(s) > 1$. Then we have:

$$F_s(a,b) = \frac{\Gamma(s-1)}{\Gamma(s)} \pi v^{2(1-s)} = \frac{\pi v^{2-2s}}{s-1}.$$

# 3 Automorphic Eigenforms

An automorphic eigenform is a function on the upper half-space S, satisfying three conditions. First, it must be invariant under the action of a certain subgroup of $G$. Second, it must be an eigenfunction of the Laplacian for the upper half-space. Finally, it must satisfy a certain growth condition. Here and throughout, let $\mathbf{K}$ denote a quadratic imaginary field over the rationals.

Let $\mathbf{a}$ denote an integral ideal in $\mathbf{O}_K$. Then, we define the subgroup:

$$\Gamma[\mathbf{a}] = \{\alpha \in SL_2(\mathbf{O}_K) \ |\alpha \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (mod\ \mathbf{a}M_2(\mathbf{O}_K))\}.$$

A subgroup $\Gamma \subset G$ is called a congruence subgroup, if $\Gamma \supset \Gamma[\mathbf{a}]$ as a subgroup of finite index for some $\mathbf{a}$. Then, a congruence subgroup $\Gamma$ a discrete subroup of $G$, with $\Gamma \backslash S$ having finite volume.

Now, with $u = x + iy$, we may compute the Laplacian on $S$, considered as the Riemannian manifold $S \cong SL_2(\mathbf{C})/SU(2)$, from the form of the invariant metric:

$$L = v^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial v^2} \right) - v\frac{\partial}{\partial v} = v^2 \left( \frac{\partial^2}{\partial v^2} + 4\frac{\partial^2}{\partial u \partial \bar{u}} \right) - v\frac{\partial}{\partial v}.$$

Then for any $\gamma \in SL_2(\mathbf{K})$ and any suitable function $f$ on $S$, since the metric is $G$-invariant:

$$[L(f)](\gamma z) = L(f(\gamma z)). \tag{12}$$

With these definitions in mind, we define an automorphic eigenform for a congruence subgroup $\Gamma$ to be a real-analytic function $f$ on $S$ satisfying the conditions:

(1) $f \circ \gamma = f$ for all $\gamma \in \Gamma$,

(2) $Lf = \lambda f$ for some $\lambda \in \mathbf{C}$,

(3) $f(\gamma(u+vj)) = O(v^\beta)$ as $v \to \infty$ for all $\gamma \in SL_2(\mathbf{K})$.

Now, we wish to show that every automorphic eigenform $f$ has an expansion in terms of K-Bessel functions. First, note that such a function $f$ is periodic with respect to the lattice $\mathbf{a}$, since it is invariant under the action of the matrices, $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ for every $b \in \mathbf{a}$. Thus, if we let $\mathbf{d}_K$ denote the different of $\mathbf{K}$, then since $\mathbf{a}^{-1}\mathbf{d}_K^{-1}$ is dual to $\mathbf{a}$ with respect to the trace map, $f$ has the expansion:

$$f(u+vj) = \sum_{a \in \mathbf{a}^{-1}\mathbf{d}_K^{-1}} c(a,v) e^{\pi i tr(au)}.$$

Since $f$ is an eigenfunction of the Laplacian, we compute $Lf$ to be:

$$Lf = \sum_{a \in \mathbf{a}^{-1}\mathbf{d}_K^{-1}} e^{\pi i tr(au)} \left[ v^2 c'' - vc' - \pi^2 N(a) v^2 c \right] = \lambda f,$$

where here we let $c'$ and $c''$ denote derivatives with respect to $v$. Hence, $c$ satisfies the differential equation:

$$v^2 c'' - vc' - (\pi^2 N(a) v^2 - \lambda) c = 0.$$

In the case $a = 0$, this yields the differential equation, $v^2 c'' - vc' + \lambda c = 0$. Letting $\alpha = 1 + \sqrt{1-\lambda}$ and $\alpha' = 1 - sqrt 1 - \lambda$ allows us to express the general solution as $v = B_1 c^\alpha + B_2 c^{\alpha'}$ for general constants $B_1$, $B_2$. In the case tha $a \neq 0$, recall that this is precisely the differential equation in (3). Moreover, the growth condition that we have stipulated gives the uniqueness of the solution, so that combining results, we get:

$$f(u+vj) = B_1 c^\alpha + B_2 c^{\alpha'} + \sum_{0 \neq a \in \mathbf{a}^{-1}\mathbf{d}_K^{-1}} c(a) v K_s(\pi |a| v) e^{\pi i tr(au)}.$$

Here, we are letting $s^2 = 1 - \lambda$.

We denote by $A(\lambda, \Gamma)$ the space of all automorphic eigenforms, for the congruence subgroup $\Gamma$ with eigenvalue $\lambda$. Furthermore, we define $S(\lambda, \Gamma)$ to be the space of cusp forms, i.e. those automorphic eigenforms with constant term zero with respect to the above expansion.

Given two elements $f, g \in A(\lambda, \Gamma)$, we define their inner product by:

$$\langle f, g\rangle = \mu(\Phi)^{-1} \int_{\Phi} \bar{f} g \, d\mu(z),$$

where we are letting $\Phi = \Gamma \backslash S$ and $\mu$ be the invariant measure that we derived previously.

## 4 Eisenstein Series for a Quadratic Imaginary Field

In this section, we define the Eisenstein Series on our space $S$ with respect to the full modular group $\Gamma = SL_2(\mathbf{O}_K)$. We will then discuss the central properties of these Eisenstein series, namely their Fourier expansion, and their functional equation. We define the Eisenstein series by:

$$E(s, z) = \eta(z)^s \sum_{0 \neq (c,d) \in \mathbf{O}_K^2} |cz + d|^{-2s}. \tag{13}$$

This is defined for $z = u + vj \in S$ and $s$ a complex number for which this series converges. More precisely, by using the integral test, we may check that the sum (13) converges absolutely for all $Re(s) > 2$. Let $f(y) = |y|^{-2s}$ for $y \in S$, and let $\hat{f}$ denote the Fourier transform of $f$ considered as a function on $\mathbf{C}$, which we computed in Section (2). Then we may evaluate the Fourier coefficients of the Eisenstein series as follows:

$$\begin{aligned} \eta(z)^{-s} E(s, z) &= \sum_{0 \neq d \in \mathbf{O}_K} |d|^{-2s} + \sum_{0 \neq c \in \mathbf{O}_K} |c|^{-2s} \sum_{a \in c^{-1}\mathbf{O}_K} f(z + a) \\ &= w\zeta_{\mathbf{K}}(s) + \sum_{0 \neq c \in \mathbf{O}_K} |c|^{-2s} Vol(\mathbf{C}/c^{-1}\mathbf{O}_K)^{-1} \sum_{b \in c\mathbf{d}_K{}^{-1}} \hat{f}(b) e^{\pi i tr(\bar{b}u)}, \end{aligned}$$

where the last step follows from the Poisson summation formula, and $w$ denotes the number of roots of unity in $\mathbf{K}$. Now we note that: $Vol(\mathbf{C}/c^{-1}\mathbf{O}_K) = \frac{1}{2}\sqrt{|D_{\mathbf{K}}|} N(c)$, so that using our formula for $\hat{f}$ from Section 2, we get:

$$\begin{aligned} v^{-s} E(s, u + vj) &= w\zeta_{\mathbf{K}}(s) \sum_{0 \neq c \in \mathbf{O}_K} \left(\frac{1}{2}\sqrt{|D_{\mathbf{K}}|}\right)^{-1} N(c)^{1-s} \\ &\quad \left( \frac{\pi v^{2-2s}}{s-1} + \sum_{0 \neq b \in c\mathbf{d}_K{}^{-1}} \frac{\pi^s}{\Gamma(s)} \left|\frac{b}{v}\right|^{s-1} K(s-1, \pi v |b|) e^{\pi i tr(u\bar{b})} \right). \end{aligned}$$

After some more computation, we arrive at the final expression:

$$v^{-s}E(s,u+vj) = w\zeta_{\mathbf{K}}(s) + \frac{2w}{\sqrt{|D_{\mathbf{K}}|}}\frac{\pi v^{2-2s}}{s-1}\zeta_{\mathbf{K}}(s-1) \quad (14)$$

$$+\frac{2}{\sqrt{|D_{\mathbf{K}}|}}\frac{\pi^s}{\Gamma(s)v^{s-1}}\sum_{0\neq b\in\mathbf{d}_K^{-1}}|b|^{s-1}e^{\pi itr(\bar{b}u)}K(s-1,\pi v|b|)\sum_{c|b\mathbf{d}_K}N(c)^{1-s} \quad (15)$$

Now, with this expansion, we may derive the desired properties of the Eisenstein Series. Namely, they are examples of automorphic eigenforms, and are orthogonal to the cusp forms with respect to our inner product. Thus we begin with

**Proposition 6** *Fixing $s$ with $Re(s) > 2$, the Eisenstein series $E(s,z)$ is an automorphic eigenform with respect to the full modular group $\Gamma = SL_2(\mathbf{O}_K)$.*

PROOF: We must check three properties: First, that it is invariant under the action of the modular group, second, that it is an eigenfunction of the Laplacian, third, that it satisfies the growth condition. It is easy to check the first of these, for elements of $\Gamma$ represent precisely those transformations that fix the lattice $\mathbf{O}_K$. Thus from the definition of the Eisenstein series in equation (13), we see that it is invariant under the action of $\Gamma$.

Now we show that it is an eigenfunction of the Laplacian. Let $\eta(z)$ denote the j component of $z$ for all $z \in S$ as before. Then we compute the Laplacian of $\eta(z)^p$:

$$\begin{aligned} L\eta(u+vj)^p &= v^2\left(\frac{\partial^2 v^p}{\partial v^2} + 4\frac{\partial^2 v^p}{\partial u \partial \bar{u}}\right) - v\frac{\partial v^p}{\partial v} \\ &= p(p-2)v^p. \end{aligned}$$

Taking $P$ to be the parabolic subgroup of $G$ as defined just before Proposition 3, we may rewrite the Eisenstein series by

$$E(s,z) = \eta(z)^s \sum_{0\neq(c,d)\in\mathbf{O}_K^2} |cz+d|^{-2s} = \sum_{\gamma\in(P\cap\Gamma)\backslash\Gamma} \eta(\gamma z)^{-2s}.$$

Then by equation (12), $E(s,z)$ is an eigenfunction with eigenvalue $\lambda = 4(s-s^2)$.

Finally, we must show that the Eisenstein series satisfies our growth condition. Namely we must check that as $v \to \infty$, $E(s,u+vj)$ is bounded by some power of $v$. However, this follows from its expansion in K-Bessel functions (15). For since $K(s,v)$ decays rapidly as $v \to \infty$, asymptotically $E(s, u+vj)$ is determined by it's constant term. But this, we have seen, involves only polynomial growth in $v$. Thus $E(s, u+vj)$ is an automorphic eigenform.

Q.E.D.

Now we show that in addition to being in the space $A(\lambda, \Gamma)$ of automorphic eigenforms, the Eisenstein series lie in a space orthogonal to the cusp forms. To see this, let $A = \mathbf{C}/\mathbf{O}_K$ and $B = \{y \mid y > 0\}$, so that the $(P \cap \Gamma)\backslash S$ is the product of $A$ and $B$. Then we may compute, given a cusp form $f$:

$$\int_A \int_B \overline{f(u+vj)} \sum_{\gamma \in P\backslash\Gamma} \eta(\gamma(u+vj))^{-2s} \frac{du d\bar{u} dv}{2iv^3}$$
$$= \sum_{\gamma \in P\backslash\Gamma} \int_B \int_A \overline{f(u+vj)} \frac{du d\bar{u}}{2i} v^{-2s+3} dv,$$

by invariance of $f$ as well as the measure $d\mu$ under transformation by $\gamma \in \Gamma$. But this last integral is clearly zero, since the constant term of $f$ is zero. Moreover, the inner product, $\langle f(z), E(s,z)\rangle$ is just a constant multiple of this integral, so we have shown that $\langle f(z), E(s,z)\rangle = 0$. Hence the Eisenstein series are orthogonal to the cusp forms.

The final property of the Eisenstein series that we discuss is its meromorphic continuation and functional equation. We have:

**Proposition 7** *The function E(s,z) extends to a meromorphic function in $s$ on all of $\mathbf{C}$, whose only possible poles are at $s = 2$ and $s = 0$. More precisely, if we let*

$$D(s,z) = \Gamma(s)\pi^{-s}E(s,z),$$

*then $D$ can be continued to a meromorphic function on $\mathbf{C}$ and $D$ satisfies the functional equation: $D(s,z) = D(2-s,z)$.*

Proof: We check the meromorphic continuation and functional equation by checking them term by term using the expansion (15). We may write the terms of the expansion of $D(s, u+vj)$ beginning with the constant term:

$$a_0 = w\Gamma(s)\pi^{-s}v^s\zeta_{\mathbf{K}}(s) + \frac{2w}{\sqrt{|D_{\mathbf{K}}|}} \frac{\pi^{1-s}v^{2-s}}{s-1}\Gamma(s)\zeta_{\mathbf{K}}(s-1), \qquad (16)$$

and the further terms:

$$a_b = \frac{2v}{\sqrt{|D_{\mathbf{K}}|}} |b|^{s-1} K(s-1, \pi v|b|) \sum_{c|b\mathbf{d}_K} N(c)^{1-s}. \qquad (17)$$

For the meromorphic continuation, we know that the constant term has a meromorphic continuation derived from that of the zeta function. Namely, recall that if we let $\xi(s) = 2^{-s} D_{\mathbf{K}}^{\frac{s}{2}} \pi^{-s} \Gamma(s) \zeta_{\mathbf{K}}(s)$, then $\xi(s)$ has an meromorphic continuation to all of $\mathbf{C}$, whose only poles are simple poles at $s = 0$ and $s = 1$. Furthermore, each non-constant term is an entire function, since the K-Bessel function is. Finally, since the sum, $\sum_{c|b\mathbf{d}_K} N(c)^{1-s}$ has only polynomial growth in $b$, while the K-Bessel function satisfies $|K(s-1, \pi v|b|)| \leq Ce^{-2\pi v|b|}$ with $s$ in any compact set, the series (15) converges for all $s$. This yields the meromorphic continuation of $D(s, z)$. The only possible poles come from the poles in the constant term, at $s = 0$ and $s = 2$, since the poles at $s = 1$ cancel each other.

Now let us derive the functional equation. We will restrict ourselves to the case when the class number of $\mathbf{K}$ is 1, though the functional equation holds for arbitrary class number. Again, we verify this term by term. For the constant term, letting $\xi(s)$ be as before, recall that $\xi(s) = \xi(1-s)$. Hence by (16), we may see that the constant term satisfies the desired functional equation. For the general term in (17), we first apply the identity (5) to the K-Bessel function that is present. Now, note that since the class number of $\mathbf{K}$ is 1, we also have the identity:

$$\begin{aligned} |b|^r \sum_{c|b\mathbf{d}_K} |c|^{-2r} &= \sum_{c \cdot g = b\sqrt{D_{\mathbf{K}}}} g^r c^{-r} \\ &= |b|^{-r} \sum_{c|b\mathbf{d}_K} |c|^{2r}. \end{aligned}$$

Applying this with $r = s - 1$ finishes the functional equation for the general term (17). Thus we have derived the entire functional equation, $D(s, z) = D(2-s, z)$.

Q.E.D.