December 14, 2007

Dear Weissman,

For me, the aim is to understand "metaplectic" forms on semi-simple groups, the hope being that they are not "new" object, but rather correspond to usual automorphic forms on some other groups, on which they give new information. I would like to have precise conjectures on the hoped for correspondence, and I view my paper [4] with Brylinski as setting a landscape in which conjectures should fit. As you do, I hope there are conjectures valid for all reductive groups, and that the case of tori gives us useful constraints on what it makes sense to conjecture. Additional constraints should be taken into account:

(1) There should be compatible global and local stories.

(2) There should be a compatibility with (global or local) fields extensions. The fact that the description [4] is functorial in the field matters here.

(3) There should be a geometric counterpart to the tame local or char. p global story: "geometric Langlands program".

(4) Existing examples are to fit in. To those you mention, I add Pattrson (on cubic Gauss sums) and Lysenko (Ann. ENS **39** 3 (2006)) on a geometric metaplectic case.

Langlands [10] suggests to me that for a general torus, $T(\mathbb{A})/T(F)$ is not the best thing to consider: the story simplifies if one considers $[T(\mathbb{A}')/T(F')]^\Gamma$, for F'/F a Galois extension with group Γ which splits T. The result does not depend on F', and for F a global field of char. p with field of constants $\mathbb{F}_q$, it is the group of $\mathbb{F}_q$-points of a group scheme over $\mathbb{F}_q$, which fits with geometric Langlands. Automorphic representations should be defined as corresponding to quasi-characters of this $[T(\mathbb{A}')/T(F')]^\Gamma$, which I will write $T(\mathbb{A}/F)$.

I dislike Langlands cocycle picture of the dual group as a semi-direct product. For me, the correct picture is the following. For G reductive over a (reasonable) scheme S, the dual

$G^\wedge$ is a local system, over S_{et}, of split pinned reductive groups over $\mathrm{Spec}(\mathbb{Z})$. This means the data, locally on S for the etale topology, for U/S etale connected of $\widehat{G}_{[U]}$ split pinned $/\mathrm{Spec}(\mathbb{Z})$, functorial in U (in the category of split pinned groups and isomorphisms of such).

For usual automorphic forms, the $\widehat{G}$ might be better replaced by $\widehat{G} \otimes_{\mathbb{Z}} \mathbb{C}$ (local system on S_{et} of complex reductive groups). For the purely algebraic story of ($\bar{\mathbb{Q}}$-valued) automorphic forms on G/function field, one may prefer $\widehat{G} \otimes_{\mathbb{Z}} \bar{\mathbb{Q}}$.

About what you do. I don't see how your local classification could lead to meaningful global conjectures, and I dislike that you choose an uniformizing parameter.

Let me consider the simplest case: over F, $\mathbb{G}_m$, the central extension by K_2 given by the cocycle $\{x, y\}$, and $n = 2$. I exclude equal characteristic 2. The central extension considered has F^* as group of automorphisms: $\mathrm{Hom}(\mathbb{G}_m, K_2)$ over F. What has a nice answer is:

$*$ For F local (resp. global), the orbits of the automorphism group F^* of the extension of $\mathbb{G}_m$ by K_2 on the set of irreducible genuine representations of the corresponding central extension fo F^* by μ_2 (resp. of genuine automorphic representations).

The central extension of F^* (resp. $\mathbb{A}^*$) by μ_2 is abelian. We are hence considering orbits of F^* acting on genuine quasi-characters of the extension of F^* by μ_2 (resp. of (extension of $\mathbb{A}^*$ by $\mu_2)/F^*$). The action is: multiplication by $(f, *)_2$. The $(f, *)_2$ are all characters of order 2 of F^* (resp. of $\mathbb{A}^*/F^*$). Orbits are hence given by the restriction to the inverse image of F^{*^2} (resp. $(\mathbb{A}^*/F^*)^2$). On the squares, the extension canonically splits. The classification is hence by characters of F^{*^2} (resp. $(\mathbb{A}^*/f^*)^2$). Equivalently: <u>by the characters of</u> F^* (resp. $\mathbb{A}^*/F^*$) <u>which are squares</u> [by $\chi \mapsto \chi \circ$ (squaring: $F^* \to F^{*^2}$ or $(\mathbb{A}^*/F^*) \to (\quad)^2$)].

This is coarser than what you do, but the only thing so far I am able to make sense of globally as well as locally.

Here is another fact which puzzles me. Let me be tame, and for simplicity consider $F = \mathbb{F}_q((t))$. Then, the central extension of F^* by $\mathbb{F}_q^*$ given by the tame symbol as cocycle is not the best one can do. Let $\mathcal{P}$ be the commutative Picard category of mod 2 graded lines on $\mathbb{F}_q$. It is not strictly commutative: $L \otimes L \to L \otimes L$ is -1 for L of degree 1. For G a group and Q a Picard category, one can speak of a central extension of G by Q: $g \in G \mapsto Q_g$ with $Q_{gh} \sim Q_g Q_h$ compatible with associativity. For Q commutative and G commutative, we get a "commutator": $G \times G \to \pi_1(Q)$:

$$Q_g Q_h = Q_{gh} = Q_{hg} = Q_h Q_g \to Q_g Q_h.$$

There is a canonical central extension of $\mathbb{F}_q(t)^*$ by $\mathcal{P}$, whose commutator is the tame symbol. Should one not also consider corresponding automorphic forms? This story works well for global fields of char. p [central extension of $\mathbb{A}^*$ by $\mathcal{P}$ trivialized over F^*], and is geometric.

I hope that some light might come from the following question.

Consider G reductive over $k[\![t]\!]$, k algebraically closed. Consider the Ind k-scheme $G(k(\!(t)\!))/G(k[\![t]\!])$ (the affine grassmannian). Extensions of G by K_2 give rise to extensions of $G(k(\!(t)\!))$ by k^*, split over $G(k[\![t]\!])$, hence to a line bundle L on the affine grassmannian, $G(k(\!(t)\!))$-equivariant. Fix an integer n and $\varepsilon\colon\ \mu_n(k) \hookrightarrow \bar{\mathbb{Q}}_\ell{}^*$. We can then consider (L, ε)-twisted $G(k[\![t]\!])$-equivariant perverse sheaves on the affine grassmannian. "Twisted" means it is really perverse sheaves on L-(0-section) (shifted by 1: $[-1]$) with monodromy ε around the 0 section. On this category, we have a convolution functor, which is the geometric counterpart of product in the (metaplectic) Hecke algebra. Arguments of Beilinson and Drinfeld should turn it into an associative commutative tensor functor. Question: Do we get a category of representations of a ("dual") group over $\mathbb{Q}_\ell$? Or of a super group? Which one?

This could be more precise than trying to match a Hecke algebra with a representation ring. If we start with G over a curve, we hopefully get a local system of tannakian categories. Whether it should come from a local system of split pinned groups over $\bar{\mathbb{Q}}_\ell$is not clear.

Lysenko addresses this question for the symplectic group. I quote a question I had for him:

> A question: Let G be split reductive over $k((t))$. If Y is the cocharacter group of a split maximal torus, a Weyl gorup invariant integral quadratic from

Q on Y determines an (isomorphism class of) extensions of G, viewed as infinite dimensional over k, by the multiplicative group. For any D, one can then repeat your definition of genuine spherical sheaves, using the gerb of Dth roots of a corresponding line bundle over the affine grassmannian. I would hope that again one gets a tannakian category. The simple objects would this time be indexed by Y_1/W, where Y_1 is the following sublattice of Y: for $B(x,y) = Q(x+y) - Q(x) - Q(y)$ the bilinear form associated to Q,

$$Y_1 = \{y \text{ in } Y | B(y, Y) \subset D\mathbb{Z}\}.$$

The story should depend only on the rational valued quadratic form Q/D, and the group corresponding ot the tannakian category should have Y_1 as weight lattice and W as Weyl group. Inspired by what you obtain for Sp (with $Q(\text{short root}) = 1$, $D = 2$ and $Y_1 = Y$) one can make a guess as to what the root system $E_1 \subset Y$ should be:

$$R_1 = \{\alpha. \text{ (denominator of } Q(\alpha)/D) | \alpha \text{ in } Y \text{ a coroot}\}?$$

Could you tell me how much of this has been investigated? Please warn me if I delude myself.

Here are some comments on your text:

Line 6–11 of 0. Main Results: I am not convinced by the usefulness of this panoply. For me, the useful category should be that of [4], and one should remember the functoriality in F. For the category of [4], π_0 = quadratic forms on Y, $\pi_1 = \mathrm{Hom}(Y, F^*)$, and it is incarnated by

$$Z \to X \otimes X$$

with Z deduced from $X \otimes X$ by pushing:

$$\begin{array}{ccc} \Gamma^2(X) & \longrightarrow & X \otimes X \\ \downarrow & & \downarrow \\ X \otimes F^* & \longrightarrow & Z \end{array}$$

with first vertical $\gamma^2(x)$ [$= x \otimes x$ in Γ^2 = symmetric tensors] mapping to $x \otimes (-1)$.

§1

1.2 I hope the definition 1.2 is not mine. One should require “non empty”, and then the unit follows.

1.5 “intertwines” should be a datum.

1.6 It is distasteful to me to view as a category what is a 2-category. I like all definitions to be such that a category should matter only up to equivalence unique up to unique isomorphism. Here, homotopies give ismorphisms of functors.

line before 1.5: Tell $\{u, u\} = \{u, -1\}$?

3 lines before 1.13: Tell your sign convention for $(x, y)_n$.

1.13: Tell simply that $\{\pi, \pi\}_n = (-1)^{(q-1)/n}$.

5.2 Why don't you just define Y_σ by this formula?

7.2 The proof seems nonsense. You rather want to check that the commutator map, which, as $\widetilde{T}'_F$ is a centralizor, is defined on the image of $\widetilde{T}'_F$ in T_F/T_F^0, is trival there.

§8. The topological $K_2(\mathbb{C})$ is trivial, and neither it nor its algebraic analog will help you with the topological $K_2(\mathbb{R})$ $(= \mu_2)$. I am sceptical of what you tell. Please look at [4] 12.6, 12.7.

All the best,

P. Deligne

E-mail: weissman@ucsc.edu

January 20, 2008

Pierre Deligne
Institute for Advanced Study

Dear Deligne,

Thank you very much for taking a close look at my work, and for your generosity in ideas.

Regarding your comment that metaplectic forms should "correspond to usual automorphic forms on some other groups": I have the following hope related to "metaplectic functoriality". The parameterization in my paper arises (after unavoidable choice of "pseudo-trivial" base point) from a two-term complex of complex dual tori $\hat{\mathcal{T}} \rightarrow \hat{\mathcal{T}}^{\#}$, which depends on the original data defining the metaplectic extension. Given a morphism of two-term complexes, i.e. a commutative diagram:

$$\begin{tikzcd} \hat{\mathcal{T}}_1 \arrow[r] \arrow[d] & \hat{\mathcal{T}}_1^{\#} \arrow[d] \\ \hat{\mathcal{T}}_2 \arrow[r] & \hat{\mathcal{T}}_2^{\#} \end{tikzcd},$$

one can lift packets of genuine irreducible representations of $\tilde{T}_1$ to packets of such representations of $\tilde{T}_2$. When one of these two-term complexes is an identity map (corresponding to an ordinary torus $\mathbf{T}_2$), one finds lifting of packets of genuine irreducible representations of $\tilde{T}_1$ to ordinary quasicharacters of T_2.

Perhaps something like this can be generalized to other groups; one could consider the Langlands dual group, endowed with a suitable isogeny of tori.

Regarding your comment "There should be compatible global and local stories": I believe that there is a global result (for split tori) that is compatible with my local result. By taking care of the "local story" for tame covers of unramified tori, I have aimed to treat "almost all" local stories for a given global metaplectic torus. In Remark 4.9 of the new draft, I describe a global result for split tori.

Regarding your suggestion, following Langlands, to consider automorphic representations of a torus as quasicharacters of $[\mathbf{T}(\mathbb{A}_L)/\mathbf{T}(L)]^\Gamma$, where L is a splitting field of $\mathbf{T}$: This has certainly been a helpful point of view, and one which I had not appreciated before. However, I am still unsure of what the correct notion of an "automorphic representation" is, for general global metaplectic tori. For split tori, I believe I understand the correct notion, but for nonsplit tori, I am having difficulty. This is partly why Remark 4.9 is limited to split tori.

Regarding your comment "I dislike Langlands cocycle picture of the dual group", I believe that I understand your dislike. I have chosen what I believe is a "middle-ground" between your suggestion of viewing the dual group as a local system, and Langlands point of view. Namely, I have chosen to define the dual torus as a torus over $\mathbb{Z}$, endowed with Galois action – I believe that this differs from a local system only in the language used (I also work over a base field rather than a more arbitrary scheme).

Regarding your comment "I dislike that you choose a uniformizing parameter", I share your distaste for noncanonical choices. However, I do not think I am able to cleanse the paper completely of this choice, though a more skilled technician would probably have the ability. I would be happy especially if Section 5 of my paper could be made cleaner and more canonical, but I have not yet succeeded in this endeavor. On the other hand, the results do not depend in any way on the choice of uniformizing element. So hopefully, the choice of uniformizing parameter will not spoil the paper too much.

Regarding your example of the simplest case, of an extension of $F^\times$ (or $\mathbb{A}^\times$) by μ_2, I believe that I have successfully generalized this example to arbitrary extensions of split tori by $\mathbf{K}_2$, both locally and globally.

Regarding the "geometric Langlands" analogues: I would like to tackle these problems in a future paper. I am especially interested in your remark/question about whether a category agrees with the category of representations of a super-group. When I began to study metaplectic tori, I noticed that the representations of the analogue of a spherical Hecke algebra corresponded to representations of a quantum dual torus (quantized at a certain root of unity). I have placed this observation in Remark 6.9. In many situations, this would involve quantization at -1, and may be equivalent to looking at a super-group.

Regarding your specific comments on my paper, I am very thankful for the advice. I have completely removed the "panoply" involving a functorial construction of split metaplectic tori, and chosen instead to follow the approach suggested by your paper with Brylinski. I am hopeful that your comments have considerably improved the quality of my paper.

Following Section 12.11 of your paper with Brylinski, I consider an extension of tori in Section 6.1 of my paper. In Remark 6.1, I wonder whether the construction of your paper could be related to an extension of Γ-modules constructed more directly. Perhaps you have thought about this as well, as it related to Question 12.13 of your paper.

I was delighted to receive the previous feedback on my work, and I would welcome any other comments you might offer.

Sincerely yours,

Martin Weissman

August 24, 2011

Dear Weissman,

While I don't like the classical description of parameters in term of the semi-direct product $G^\vee \rtimes \mathrm{Gal}$, I found your wish to construct a non trivial extension of Gal by $\widetilde{G}^\vee$ very interesting. Long ago, when I asked Langlands "why the semi-direct product?", he answered "what else can you define?". Here you can!

I don't like Hopf algebras, which for me hide the geometry (meaning either the group scheme, or the category of representations), and I hope one can get rid of them. You did it for the first twist: one has just a cocycle with values in elements of order 2 of the center of $\widetilde{G}^\vee$. I did not understand your second twist yet, in part because you use works of Lusztig I am not mastering. I would like to understand the constructed extension

$$\widetilde{G}^\vee \to * \to \Gamma$$

by first understanding the inverse image of $\gamma \in \Gamma$ as a right and left principal homogeneous space under $\widetilde{G}^\vee$. First I need to make sure that $\widetilde{G}^\vee$ is what I think it is: it is given with $\widetilde{T}^\vee \subset \widetilde{B}^\vee \subset \widetilde{G}^\vee$, and a pinning (generators of the $\mathrm{Lie}(\widetilde{G}^\vee)_\alpha$ for α a smple root; in french: épinglage), this defining it up to unique isomorphism. Your text does not mention the pinning, which makes me worry.

Here is how I like to think to the classical $G^\vee$ and G^L. Suppose G is a reductive group over a scheme S. Locally for the etale topology on S, G is split, and the meaning of $G^\vee$ is clear. We get $G^\vee$ to be a local system, on the etale site S_{et} of S, of pinned split reductive gorups over $\mathbb{Z}$. Even for $S = \mathrm{Spec}(F)$, F a field, I prefer to speak in term of sheaves over $\mathrm{Spec}(F)_{\mathrm{et}}$ rather than to choose a separable closure $\bar{F}$ of F and use actions of $\mathrm{Gal}(\bar{F}/F)$. For instance, one can consider l-adic sheaves on $\mathrm{Spec}(F)$. Once $\bar{F}$ is chosen, it is the same as l-adic representations of $\mathrm{Gal}(\bar{F}/F)$, but the notion of l-adic sheaf does not require the choice of a $\bar{F}$. Similarly, here is how to view morphisms

$$\mathrm{Gal}(\bar{F}/F) \to G^L(\overline{\mathbb{Q}}_l) \tag{1}$$

projecting identically to the Gal quotient of G^L.

Let us say that an etale extension F' of F is large enough if on F' G becomes an inner form of a split group (= the connected component of the center is a split torus, and the action of $\mathrm{Gal}(\bar{F}'/F')$ on the Dynkin diagram is trivial. This action is the action of $\mathrm{Gal}(\bar{F}'/F)$ on the set of conjugacy classes of maximal parabolics of $G \otimes_{F'} \bar{F}'$). If F'/F is large enough, the dual $G^\vee[F']$ of $G \otimes_F F'$ is defined. It is a split pinned reductive group over $\mathbb{Z}$, and for $F' \to F''$ a morphism between large enough etale extensions of F, we have a canonical isomorphism $G^\vee[F'] \xrightarrow{\sim} G^\vee[F'']$. This is the sense in which $G^\vee$ is a local system on $\mathrm{Spec}(F)_{\mathrm{et}}$.

One can now view a map (1) as the same thing as the following: the data, functorial in F'/F large enough and in a representation V of $G^\vee[F']$ over $\overline{\mathbb{Q}}_l$, of a $\overline{\mathbb{Q}}_l$-sheaf $\mathcal{F}(F', V)$ on $Spec(F')$, plus a compatibility data

$$\mathcal{F}(F', V_1) \otimes \mathcal{F}(F', V_2) \xrightarrow{\sim} \mathcal{F}(F', V_1 \otimes V_2) \tag{2}$$

Requested: $\mathcal{F}$ is an exact tensor functor in V, and $\mathcal{F}(F', V) \xrightarrow{\sim} \mathcal{F}(F'', V)$ for F''/F' (compatible with (2)).

What I like in that description is that there was no need to first choose $\bar{F}$, and that one can replace in it "$\overline{\mathbb{Q}}_l$-sheaf" by objects of a de Rham flavor, or try to replace it by "motives". For dR, one will be led to consider the $G^\vee[F'] \otimes_{\mathbb{Z}} F'$: this is a reductrive group given locally over $\mathrm{Spec}(F)_{\mathrm{et}}$, and descends to a reductive group $G^\vee_F$ over F, given with a maximal torus, a Borel containing it, and a pinning.

Note that we did not really need $G^\vee[F']$, but only its tensor category of representations, and that the latter depends only on the (trivial) gerb of $G^\vee[F']$-torsors: a representation is a way to attach functorially to a $G^\vee$-torsor a vector space [I would need the $fppf$ site here to be correct].

An extension

$$\widetilde{G}^\vee \to * \to \Gamma$$

defines such a gerb, hence your parameter would also have an interpretation as above.

To help me guess what your second twist is, could you tell me what the trists are for G a torus.

A few local comments:

Page 7 Construction 1.3. It might be worth observing that the construction depends only on the quadratic form $\frac{1}{n} Q$ on Y with values in $\mathbb{Q}$, not separately on n and Q. This is clearer if you say: n_i denominator of $\frac{1}{n} Q(\alpha_i^\vee)$, and $\frac{1}{n} B(y, y') \in \mathbb{Z}$.

Page 16, line -4. You might mention that $h(\gamma_1, \gamma_2)$ gives the action of γ_1 on the n^{th} roots of $\mathrm{rec}(\gamma_2)$: Hilb is the comparison of class field and Kummer theories. In 3.11, η gives the action of γ on n^{th} roots of -1.

Construction 2.5 "essentially surjective": I would prefer: "a functor from ... to ... which is surjective on isomorphism lasses of objects", if this is what you mean.

Before Proposition 3.17: Add: "This is a short exact sequence in the following sense:"? [It is also a short exact sequence of *fppf* sheaves on $\mathrm{Spec}(\mathbb{Z})$.]

You are extremely cautious about viewing an ordinary finite group as a group scheme (over $\mathrm{Spec}\,\mathbb{Z}$ or, by base change, over anything). I like to think to it as follows: a set E defines a scheme $\underline{E}$ over $\mathrm{Spec}(\mathbb{Z})$: the disjoint sum of copies of $\mathrm{Spec}(\mathbb{Z})$ indexed by E. It is affine when E is finite. The functor $E \mapsto \underline{E}$ is left adjoint to "$\mathbb{Z}$-points of". It is compatible with finite limits or colimits, hence transforms groups into groups. If schemes over $\mathrm{Spec}(\mathbb{Z})$ are viewed as sheaves over some big site, it is $E \mapsto$ constant sheaf E.

Another thing I dislike about the classical story is that unitary induction is used, and that it makes the story irrational (= not defined over $\mathbb{Q}$: some $\sqrt{p}$ are introduced). To exorcise this, I was led to consider in a reductive group G the canonical central element of order 2 which over the algebraic closure can be described as $\Pi\, \alpha^\vee(-1)$ (product over positive roots), and dually to consider a canonical double covering. It does not seem related to what you do but, if I can find them, I will send you my notes (= a letter to Serre) anyway.

Best,

P. Deligne

Dept. of Mathematics, UC Santa Cruz

October 14, 2011

Pierre Deligne
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Dear Deligne,

Thank you for your advice sincere feedback on my work. First, I must apologize for the tardy reply. Since your letter in August, I have married, traveled to Vietnam on honeymoon, and I am teaching this semester. Life has been happy and busy. I hope I have not forgotten too much in the past two months. I'm excited about the ideas you've mentioned, and I've attempted to learn more about gerbes and Tannakian categories. I think I can answer some of your questions and concerns.

Regarding Hopf algebras, I share your distaste. They seem to me most useful for checking identities mechanically, not for gaining intuition. Especially for checking compatibility between the two twists (checking that the twisted comultiplication and twisted multiplication formed a bialgebra), I knew that some mechanical work would be involved. The Hopf algebra approach seemed transparent for a referee to check, at least.

Regarding your concern about the pinning, I believe that the construction using Lusztig's canonical basis gives a group scheme over $\mathbb{Z}$ with a pinning. The torus and Borel are constructed along with the group scheme, and the generators of the Lie algebras of the root subgroups are elements of the canonical basis of Lusztig. Perhaps this should be mentioned explicitly in my work.

I appreciate your detailed explanation of why it is better to work with local systems on the étale site over F rather than semidirect products. This is something you mentioned in an earlier letter (14 December 2007) but which I did not fully understand at the time.

Now, I will attempt to translate my double-twist into the framework you mention. Let F be a local field (not isomorphic to $\mathbb{C}$), and F_{et} the étale site. Let $\mathbf{G}$ be a connected reductive group over F with root datum $\Phi = (X, Y, \ldots)$. Let $\mathbf{G}'$ be a central extension of $\mathbf{G}$ by $\mathbf{K}_2$, over F, as discussed and classified in your paper with Brylinski. Let n be a positive integer, such that the equation $\zeta^n = 1$ has n solutions in F. Let $\epsilon : \mu_n(F) \to \bar{\mathbb{Q}}_\ell^\times$ be a homomorphism. Suppose, for simplicity, that $\mathbf{G}$ is **split** over F.

In this case the dual group $\mathbf{G}^\vee$ makes sense as a split pinned reductive group over $\mathbb{Z}$. Let $Sh(F_{et}, \bar{\mathbb{Q}}_\ell)$ be the tensor category of $\bar{\mathbb{Q}}_\ell$-sheaves on F_{et}. Let $Rep(\mathbf{G}^\vee, \bar{\mathbb{Q}}_\ell)$ be the tensor category of representations of $\mathbf{G}^\vee$ over $\bar{\mathbb{Q}}_\ell$.

Then, if I understand your suggestion in the split case, one can view a morphism from $Gal(\bar{F}/F)$ to the L-group, lying over $Gal(\bar{F}/F)$, instead as an exact tensor functor:

$$(\rho, r) : Rep(\mathbf{G}^\vee, \bar{\mathbb{Q}}_\ell) \to Sh(F_{et}, \bar{\mathbb{Q}}_\ell).$$

Here ρ denotes the $\bar{\mathbb{Q}}_\ell$-linear functor on the underlying $\bar{\mathbb{Q}}_\ell$-linear abelian category, and r denotes the functorial system of isomorphisms:

$$r : \rho(V_1) \otimes \rho(V_2) \to \rho(V_1 \otimes V_2).$$

The data (ρ, r) replace a "$\mathbf{G}^\vee(\bar{\mathbb{Q}}_\ell)$-valued Galois representation" in this case.

Now, I will attempt to incorporate the metaplectic data.

First, the central extension of $\mathbf{G}$ by $\mathbf{K}_2$ defines, in a functorial way, a central extension

$$F^\times \to E \to Y.$$

Second, the central extension of $\mathbf{G}$ by $\mathbf{K}_2$ defines a Weyl-invariant quadratic form $Q : Y \to \mathbb{Z}$, with associated bilinear form $B : Y \times Y \to \mathbb{Z}$.

In my paper, following work of Finkelberg-Lysenko, Reich, and others, I explain how the quadratic form Q can be used to modify the root datum Φ; let $\tilde{\Phi}^\vee$ be the modified dual root datum. In this modification, Y is replaced by

$$\tilde{Y} = \{y \in Y : B(y, y') \in n\mathbb{Z} \text{ for all } y' \in Y\}.$$

The other details are found in my paper. This modified root datum yields a split pinned reductive group $\tilde{\mathbf{G}}^\vee$ over $\mathbb{Z}$.

Let $C : Y \times Y \to F^\times$ be a 2-cocycle incarnating the central extension E of Y by $F^\times$. Then for each $y_1, y_2 \in Y$, we obtain a $\bar{\mathbb{Q}}_\ell$-sheaf on F_{et} as follows: for each finite Galois extension F'/F in which $C(y_1, y_2)$ is an n^{th} power, choose an element $x_{F'} \in F'$ such that $x_{F'}^n = C(y_1, y_2)$. For each F-algebra homomorphism $\gamma : F' \to F''$ of such Galois extensions, define

$$c_\gamma(y_1, y_2) = \epsilon\left(\frac{\gamma(x_{F'}^n)}{x_{F''}}\right) \in \mu_n(\bar{\mathbb{Q}}_\ell).$$

The data $c_\gamma(y_1, y_2)$ gives an $\bar{\mathbb{Q}}_\ell$-sheaf $H(y_1, y_2)$on F_{et}.

For all $y_1, y_2 \in \tilde{Y}$, we have $C(y_1, y_2)/C(y_2, y_1) = (-1)^{B(y_1, y_2)} \in (-1)^{n\mathbb{Z}}$. This, I think, determines an isomorphism of ℓ-adic sheaves:

$$h(y_1, y_2) : H(y_1, y_2) \to H(y_2, y_1).$$

The cocycle condition, $C \in Z^2(Y, F^\times)$, determines an isomorphism

$$h(y_1, y_2, y_3) : H(y_1, y_2 + y_3) \otimes H(y_2, y_3) \to H(y_1, y_2) \otimes H(y_1 + y_2, y_3).$$

It seems to me that there is a nicer way of carrying this out, as a functor (2-functor) from a category of central extensions of Y by $F^\times$ to a category of central extensions of Y by the Picard category of rank-one $\bar{\mathbb{Q}}_\ell$-sheaves on F_{et}. I leave this for a later time.

The Hilbert symbol, composed with the reciprocity isomorphism of class field theory, gives a cocycle $\kappa \in Z^2(Gal(F^{sep}/F), \mu_n)$, once a separable closure F^{sep} is chosen. I believe that, suitably rephrased, this κ gives a μ_n-gerbe (up to unique isomorphism, I hope) on F_{et}. Taking a suitable hypercovering $U_\bullet$ of F in the étale site, the cocycle κ gives elements $\kappa_{ijk} \in \mu_n(F_i \otimes_F F_j \otimes_F F_k)$ for triples F_i, F_j, F_k of sufficiently large extensions of F.

The homomorphism ϵ from $\mu_n(F)$ to $\bar{\mathbb{Q}}_\ell^\times$ allows us to consider the κ^m-twisted ℓ-adic sheaves on F_{et} for any $m \in \mathbb{Z}/n\mathbb{Z}$. Namely, an κ^m-twisted ℓ-adic sheaf on F_{et} is a family of ℓ-adic sheaves S_i on sufficiently large extensions F_i/F, and gluing isomorphisms $\sigma_{ij} : Res_{ij}(S_i) \to Res_{ij}(S_j)$ of sheaves on $F_{ij} = F_i \otimes_F F_j$, satisfying a twisted gluing condition

$$\sigma_{jk} \circ \sigma_{ij} = \epsilon(\kappa_{ijk}^m)\sigma_{ik}$$

(I learned this machinery from A.J. de Jong, "A result of Gabber". I don't know much more about twisted sheaves).

Let $Sh_m(F_{et}, \bar{\mathbb{Q}}_\ell)$ be the category of κ^m-twisted $\bar{\mathbb{Q}}_\ell$-sheaves on F_{et}. Let

$$Sh_\bullet(F_{et}, \bar{\mathbb{Q}}_\ell) = \prod_{m \in \mathbb{Z}/n\mathbb{Z}} Sh_m(F_{et}, \bar{\mathbb{Q}}_\ell).$$

Then $Sh_\bullet(F_{et}, \bar{\mathbb{Q}}_\ell)$ is a tensor category, I think, with the tensor product respecting the twist:

$$\otimes : Sh_{m_1}(F_{et}, \bar{\mathbb{Q}}_\ell) \times Sh_{m_2}(F_{et}, \bar{\mathbb{Q}}_\ell) \to Sh_{m_1+m_2}(F_{et}, \bar{\mathbb{Q}}_\ell).$$

Now, I think I can describe the modification for metaplectic groups, when $\mathbf{G} = \mathbf{T}$ is a torus. In this case, $\mathbf{T}^\vee$ is a split torus over $\mathbb{Z}$ with character lattice Y, and $\tilde{\mathbf{T}}^\vee$ is a split torus over $\mathbb{Z}$ with character lattice $\tilde{Y}$. For any $y \in \tilde{Y}$, let V_y be the standard (underlying vector space is just $\bar{\mathbb{Q}}_\ell$) irreducible representation of $\tilde{\mathbf{T}}^\vee$ over $\bar{\mathbb{Q}}_\ell$, with weight y. Thus

$$V_{y_1} \otimes V_{y_2} = V_{y_1+y_2} = V_{y_2} \otimes V_{y_1}.$$

A *metaplectic parameter* should consist of a pair (ρ, r), where

$$\rho : Rep(\tilde{\mathbf{T}}^\vee, \bar{\mathbb{Q}}_\ell) \to Sh_\bullet(F_{et}, \bar{\mathbb{Q}}_\ell)$$

is a $\bar{\mathbb{Q}}_\ell$-linear functor such that $\rho(V_y) \in Sh_{Q(y)}(F_{et}, \bar{\mathbb{Q}}_\ell)$ and $r = \{r_{y_1,y_2} : y_1, y_2 \in \tilde{Y}\}$ is a system of isomorphisms

$$r_{y_1,y_2} : \rho(V_{y_1}) \otimes \rho(V_{y_2}) \to \rho(V_{y_1+y_2}) \otimes H(y_1, y_2),$$

for all $y_1, y_2 \in \tilde{Y}$, (note that $Q(y_1+y_2) = Q(y_1)+Q(y_2)$, mod n, since $y_1, y_2 \in \tilde{Y}$) such that

1. For $y_1, y_2, y_3 \in \tilde{Y}$, the diagram

$$\begin{tikzcd}
\rho(V_{y_1}) \otimes (\rho(V_{y_2}) \otimes \rho(V_{y_3})) \arrow[r, "Id\otimes r"] \arrow[d, "assoc"] & \rho(V_{y_1}) \otimes \rho(V_{y_2+y_3}) \otimes H(y_2, y_3) \arrow[r, "r"] & \rho(V_{y_1+y_2+y_3}) \otimes H(y_1, y_2+y_3) \otimes H(y_2, y_3) \arrow[d, "Id\otimes h(y_1,y_2,y_3)"] \\
(\rho(V_{y_1}) \otimes \rho(V_{y_2})) \otimes \rho(V_{y_3}) \arrow[r, "r\otimes Id"] & \rho(V_{y_1+y_2}) \otimes \rho(V_{y_3}) \otimes H(y_1, y_2) \arrow[r, "r\otimes Id"] & \rho(V_{y_1+y_2+y_3}) \otimes H(y_1, y_2) \otimes H(y_1+y_2, y_3)
\end{tikzcd}$$

   commutes

2. For all $y_1, y_2 \in \tilde{Y}$, the diagram

$$\begin{tikzcd}
\rho(V_{y_1}) \otimes \rho(V_{y_2}) \arrow[r] \arrow[d, "comm"] & \rho(V_{y_1+y_2}) \otimes H(y_1, y_2) \arrow[d, "Id\otimes h(y_1,y_2)"] \\
\rho(V_{y_2}) \otimes \rho(V_{y_1}) \arrow[r] & \rho(V_{y_1+y_2}) \otimes H(y_2, y_1)
\end{tikzcd}$$

   commutes.

3. ρ sends the trivial representation V_0 to the constant sheaf $\bar{\mathbb{Q}}_\ell$.

These are twisted versions of the conditions for a **tensor functor**, from your paper with Milne on Tannakian categories.

Of course, I hope that these parameters (ρ, r), inspired by your perspective on Langlands parameters, will lead to a good idea for connected reductive groups, split or not. I will continue my investigations along these lines, and of course I would enjoy continued conversation.

Another direction would be to follow your remark that one can consider de Rham or motivic variants. One possibility that strikes me as interesting would be the archimedean case, even for a double cover of a torus over $\mathbb{R}$. There, I would hope that a metaplectic parameter as I've defined it corresponds to something interesting in Hodge theory.

As always, I appreciate your advice. One possibility, if you are available, would be to discuss this in person sometime in December. I finish teaching around December 5, and I plan on visiting family in New York and Washington DC shortly after. I would be happy to stop by IAS sometime if you are around.

Sincerely,

Marty Weissman

October 25, 2011

Dear Weissman,

Your construction continues to perplex me.

The first twist I hope I understand: $\frac{Q}{n}$, restricted to $\widetilde{Y}$, is with values in $\frac{1}{2}\mathbb{Z}$ and, mod $\mathbb{Z}$, is an homomorphism $\widetilde{Y} \to \frac{1}{2}\mathbb{Z}/\mathbb{Z}$ which vanishes on the $\tilde{\alpha}_1^\vee$, giving a central element of order 2, call it z, of $\widetilde{G}^\vee$. You use it to define, by a cocycle, on extension of $\mathrm{Gal}(\bar{F}/F)^{\mathrm{ab}}$ by $\widetilde{G}^\vee$: $\widetilde{G}^\vee \to E \xrightarrow{\pi} \mathrm{Gal}$.

Let me assume n even. For the second twist, at least in the case of tori, you modify the multiplication of functions on each $\pi^{-1}\gamma$, $\gamma \in \mathrm{Gal}$, using the restriction of a bisector C to $\widetilde{Y}$. If I understand correctly, this restriction is symmetric and $C(\tilde{y}_1, \tilde{y}_1)$ is hence an homomorphism $l\colon \widetilde{Y} \to \mathbb{Z}/2$ (the same as before?). One hence has $C(\tilde{y}_1, y_2) = l(\tilde{y}_1)l(\tilde{y}_2) + A$ with A alternating, hence associated to a quadratic form. If one chooses such a quadratic form, call it α, and view it as with values in ± 1, in the basis of the $\alpha(\tilde{y}_1), \tilde{y}_1$, of functions on $\pi^{-1}(\gamma)$, the new multiplication corresponds to addition of $\tilde{y}$'s, times $(-1)^{l(\tilde{y}_1)l(\tilde{y}_2)}$. We also have $\mathrm{Gal} \to \pm 1$, giving its action on n^{th} root of -1, and the modification has to be done only for the $\pi^{-1}(\gamma), \gamma \to -1$.

In the new basis, the modification can also be described as follows: our element z of order 2 acts by translations on $\pi^{-1}(\gamma)$, and for $\gamma \to -1$, this allows us to twist $\pi^{-1}(\gamma)$ by the μ_2-torsor of square roots of -1.

This kind of twist makes good sense for any G, split or not, but unfortunately does not use the extension of $Y(\text{torus})$ by $\mathbb{G}_m$ over $\mathrm{Spec}(F)$.

In the simplest case, I do not understand why this second twist: take $G = \mathbb{G}_m$, $n = 2$, extension by μ_2 defined by $(x, y)_2$, viewed as a cocycle. Hence

$$Y = \mathbb{Z},\ Q(n) = n^2,\ \widetilde{Y} = Y,$$
$$\widetilde{G}^\vee = \mathbb{G}_m \text{ again, with } z = -1$$

and (possibly I made a mistake here) the second twist occurs when -1 is not a square, above γ such that $\gamma\left(\sqrt{-1}\,\right) = -\sqrt{-1}$.

Here, the central extension of $\mathbb{G}_m$ used is commutative, and isomorphism classes of genuine representations can be identified with functions χ on F^* such that $\chi(xy) = \chi(x)\chi(y)(x, y)_2$ This fits with the central extension of $\mathrm{Gal}^{\mathrm{ab}}$ by $\widetilde{\mathbb{G}}_m^\vee$ defined by the cocycle you use. But why the second twist?

Best,

P. Deligne

P.S. Here is another perlexity. I don't know whether they are related.

If F is a non archimedean local field with residue field k, the central extension of F^* by k^* given by the tame symbol, viewed as a cocycle, has a natural "square root" for which the tame symbol is the commutator. Of course, as (x, x) is non trivial, a grain of salt is needed: it is a central extension by the commutative Picard category of mod 2 graded lines over k, with Koszul's rule for the commutativity of $\otimes$. If M is a module of finite length over the valuation ring $\mathcal{O}$ of F, one has a $\det(M)$ which is a mod 2 graded line over k, with $\det(M) \sim \det(M') \otimes \det(M'')$ for M extension of M'' by M'. Up to unique isomorphism, the central extension depends on the choice of a one-dimensional vector space V over F. If L is any lattice in V, the mod 2 graded line attached to $f \in F^*$ is $[L\colon fL] = \det(L/fL)$ if $fL \subset L$. It is independent of L: for $L'' \subset L'$, $\det(L//L'') \det(L''/L'') \det(L''/fL'') \sim \det(L'/fL') \det(fL'/fL'')$ and by $f\colon L'/L'' \xrightarrow{\sim} fL'/fL''$, one gets the independence.

A similar story holds for $\mathrm{GL}(n)$ (by restriction to $\mathrm{SL}(n)$ one gets the usual central extension), and there is a global counterpart in the function fields case.

I don't know whether such extensions have automorphic meaning, or if they give rise to a "dual".

October 26, 2011

Dear Weissman,

You seem to take for granted that the L-group should not depend on the faithful character $\varepsilon\colon \mu_n \to \overline{\mathbb{Q}}_l$ used. Is that reasonable? If we don't take it for granted, one should replace $\mathbb{Z}$ by the cyclotomic ring Λ generated by a generator $\zeta \in \mu_n(F)$ [= the quotient of the group algebra $\mathbb{Z}[\mu_n(F)]$ where $\mathrm{cyclotomic}_n(\zeta) = 0$]. In the case of a split torus, one can then do something very naive but in which I have more confidence than in your construction.

Fix T split over F, an extension of T by K_2, n, such that all n^{th} root of 1 are in F, and consider the resulting central extension E of $T(F)$ by $\mu_n(F)$. Let Q be the corresponding quadratic form on the cocharacter group Y, B the associated bilinear form, and

$$\widetilde{Y} = \left\{ \tilde{y} \in Y \mid \tfrac{B}{n}(\tilde{y}, y) \in \mathbb{Z} \text{ for all } y \in Y \right\}.$$

We have the torus $\widetilde{T} := \widetilde{Y} \otimes \mathbb{G}_m$ over F, and the center of E is the inverse image by $E \to T(F)$ of

$$\mathrm{Im}(\widetilde{T}(F) \to T(F))$$

We care about irreducible $\overline{\mathbb{Q}}_l$ representations of E for which $\mu_n(F)$ acts by $\varepsilon\colon \mu_n(F) \hookrightarrow \overline{\mathbb{Q}}_l^*$. This is the same as a character of the center Z of E, inducing ε on $\mu_n(F)$, and such a character induces a character of the pull-back $\widetilde{E}$ of E by $\widetilde{T}(F) \to T(F)$: a commutative extension

$$\mu_n \to \widetilde{E} \to \widetilde{T}(F).$$

As $\widetilde{T}(F) = \widetilde{Y} \otimes F^*$, if $\widetilde{X}$ is the dual of $\widetilde{Y}$, the data of such an extension is equivalent to the data of an extension $\widetilde{E}_1$ of E^* by $\mu_n \otimes \widetilde{X}$. On Λ, we have $\mu_n(F) = \mu_n(\Lambda) \hookrightarrow \mathbb{G}_m$, and pushing we get over Λ an extension T^L

$$\mathbb{G}_m \otimes \widetilde{X} \to T^L \to F^*$$

where F^* can now be thought of as a Weil group.

A character $\widetilde{E} \to \overline{\mathbb{Q}}_l^*$, extending ε, is the same thing as a splitting of the extension

$$\overline{\mathbb{Q}}_l^* \to * \to \widetilde{T}(F)$$

deduced from $\widetilde{E}$ by pushing by $\varepsilon\colon \mu_n(F) \to \overline{\mathbb{Q}}_l^*$. This is the same as a splitting of

$$T^L(\overline{\mathbb{Q}}_l) \to F^*,$$

where one uses $\varepsilon\colon \Lambda \to \overline{\mathbb{Q}}_l$ to give meaning to $T^L(\overline{\mathbb{Q}}_l)$.

As this construciton is very naive, it has the virtue that automorphisms of the extension of T by K_2^* we started with act on T^L, respecting the projection to F^*, and respecting the map

$$\begin{aligned}&(\varepsilon\text{-genuine irreducible } \overline{\mathbb{Q}}_l\text{-representations of } E) \to\\&(\text{Langlands parameter } W(\ /F) \to T^L)\end{aligned}$$

I hope something equally naive can be done globally. What I would like to see next is how to handle the cases of SL(2), $n = 2$ or 3, locally and globally. What is known?

Best,

P. Deligne

Dept. of Mathematics
UC Santa Cruz

November 21, 2011

Dear Deligne

First, I absolutely agree that the L-group should be defined over the cyclotomic ring $\Lambda = \mathbb{Z}[\zeta]$ as you suggest.

Regarding the construction you mention for split tori – the "something very naïve" – I am a bit embarassed that I never tried this. Of course, I'm aware of the natural isomorphism of Ext groups, $Ext(A \otimes Y, B) \cong Ext(A, B \otimes X)$, when A, B are $\mathbb{Z}$-modules and X, Y are finite-rank free $\mathbb{Z}$-modules in perfect duality. But I had never thought of the corresponding functor, from the category $\mathsf{Ext}(A \otimes Y, B)$ to the category $\mathsf{Ext}(A, B \otimes X)$. In our context, this functor must be given by the following construction (what else could it be?):

Begin with the extension of $\mathbb{Z}$-modules (recall that $\tilde{T}(F) = \tilde{Y} \otimes F^\times$)

$$\mu_n \to \tilde{E} \to \tilde{Y} \otimes F^\times. \tag{1}$$

Tensor with $\tilde{X} = Hom(\tilde{Y}, \mathbb{Z})$:

$$\mu_n \otimes \tilde{X} \to \tilde{E} \otimes \tilde{X} \to \tilde{X} \otimes \tilde{Y} \otimes F^\times. \tag{2}$$

Pull back via the canonical homomorphism $\iota : \mathbb{Z} \to End_{\mathbb{Z}}(\tilde{Y}) = \tilde{X} \otimes \tilde{Y}$:

$$\mu_n \otimes \tilde{X} \to \iota^*(\tilde{E} \otimes \tilde{X}) \to F^\times. \tag{3}$$

Now push forward as you say, via $\epsilon : \mu_n(F) \to \bar{\mathbb{Q}}_\ell^\times$:

$$\tilde{T}^\vee \to {}^L\tilde{T} \to F^\times \tag{4}$$

where $\tilde{T}^\vee = \bar{\mathbb{Q}}_\ell^\times \otimes \tilde{X}$.

I think I'm merely restating what you wrote in your previous letter, but it helped me to write out a few details I hadn't understood.

Now let me compare this to my previous perspective using messier cocycles. Suppose that the extension (1) is incarnated by a (bimultiplicative) cocycle of the form

$$c_{(1)}(y_1 \otimes f_1, y_2 \otimes f_2) = (f_1, f_2)_n^{C(y_1, y_2)},$$

where $C : Y \otimes Y \to \mathbb{Z}$ is a $\mathbb{Z}$-bilinear map such that $C(y, y) = Q(y)$, and $(\cdot, \cdot)_n$ denotes the appropriate Hilbert symbol. Recall that $Q(y_1 + y_2) = Q(y_1) + Q(y_2)$, mod n, for all $y_1, y_2 \in \tilde{Y}$.

To go further, I'll commit a sin and choose a $\mathbb{Z}$-basis $x_1, \ldots, x_r$ of $\tilde{X}$ and dual basis $y_1, \ldots, y_r$ of $\tilde{Y}$. Then the second extension is incarnated by a cocycle, $\mathbb{Z}$-bilinear in $\tilde{X}$, satisfying

$$c_{(2)}(x_i \otimes y_1 \otimes f_1, x_j \otimes y_2 \otimes f_2) = \delta_{ij} x_i (f_1, f_2)_n^{C(y_1, y_2)}.$$

The homomorphism ι sends 1 to $\sum_i x_i \otimes y_i$. Hence the third central extension is incarnated by a cocycle,

$$c_{(3)}(f_1, f_2) = c_{(3)} \left(\sum_i x_i \otimes y_i \otimes f_1, \sum_j x_j \otimes y_j \otimes f_2 \right) = \prod_i (f_1, f_2)_n^{C(y_i, y_i)} \otimes x_i.$$

So indeed, viewing this cocycle as having values in $\mu_n \otimes \tilde{X} = Hom(\tilde{Y}, \mu_n)$, it is given by

$$c_{(3)}(f_1, f_2)(y) = (f_1, f_2)_n^{Q(y)}.$$

This agrees with the "first twist" of my paper (which I am thankful for).

This implies that the L-group ${}^L\tilde{T}$ is isomorphic to the twisted product $F^\times \times_c \tilde{T}^\vee$ – the direct product of underlying sets, with multiplication twisted by the cocycle $c(f_1, f_2)(y) = (f_1, f_2)_n^{Q(y)}$. The twisted product is uniquely determined by Q, but the isomorphism is not uniquely determined by Q.

This *non-uniqueness of isomorphism* explains why something like a second twist is necessary. I think it would be incorrect to say that the L-group equals $F^\times \times_c \tilde{T}^\vee$, since it is noncanonically isomorphic. Of course, your construction avoids cocycles entirely, but maybe I can say why a second twist is natural. I think this is explained near the end of SGA 7, Exposé VII. From that source, I learned that the extension $\mu_n \to \tilde{E} \to \tilde{Y} \otimes F^\times$ can be viewed as a biextension of $(\tilde{Y}, F^\times)$ by μ_n. Using the resolutions $L_\bullet(\tilde{Y})$ and $L_\bullet(F^\times)$ of the $\mathbb{Z}$-modules $\tilde{Y}$ and $F^\times$, described in loc. cit., such a biextension, trivialized over $L_0(\tilde{Y}) \otimes L_0(F^\times) = \mathbb{Z}[\tilde{Y}] \otimes \mathbb{Z}[F^\times]$, gives a pair of functions

$$\tau : F^\times \times F^\times \times \tilde{Y} \to \mu_n, \quad \chi : F^\times \times \tilde{Y} \times \tilde{Y} \to \mu_n.$$

satisfying "cinq conditions de compatibilité". I think that τ captures the cocycle $c_{(3)}$ described above. But I think to capture the biextension structure completely, the function χ (capturing my "second twist") is also required.

Now let me try to interpret your construction from the Tannakian perspective. Begin again with the extension of $\mathbb{Z}$-modules

$$\mu_n \to \tilde{E} \to \tilde{Y} \otimes F^\times.$$

For each $y \in \tilde{Y}$, one may pull back to give an extension

$$\mu_n \to \tilde{E}_y \to F^\times.$$

This defines a homomorphism of Picard categories

$$\tilde{Y} \to \mathsf{Ext}(F^\times, \mu_n), \quad y \mapsto (\mu_n \to \tilde{E}_y \to F^\times).$$

The Hilbert symbol $Hilb_n$ gives an object $\tilde{F}^\times$ of $\mathsf{Ext}(F^\times, \mu_n)$;

$$\mu_n \to \tilde{F}^\times \to F^\times,$$

In the Picard category $\mathsf{Ext}(F^\times, \mu_n)$, $n \cdot \tilde{F}^\times = 0$ and by construction, the extension $\tilde{E}_y$ is isomorphic to $Q(y) \cdot \tilde{F}^\times$. Choose such an isomorphism $s(y) : \tilde{E}_y \to Q(y) \cdot \tilde{F}^\times$ for each $y \in \tilde{Y}$. The set of such isomorphisms is a $Hom(F^\times, \mu_n)$-torsor. There is an associated 2-cocycle $\chi : \tilde{Y} \times \tilde{Y} \to Hom(F^\times, \mu_n)$, making the following diagram commute

$$\begin{tikzcd}
E_{y_1} + E_{y_2} \arrow[r, "s(y_1)+s(y_2)"] \arrow[d] & (Q(y_1) \cdot \tilde{F}^\times) + (Q(y_2) \cdot \tilde{F}^\times) \arrow[d, "\chi(y_1,y_2)"] \\
E_{y_1+y_2} \arrow[r, "s(y_1+y_2)"] & Q(y_1+y_2) \cdot \tilde{F}^\times.
\end{tikzcd}$$

in the Picard category $\mathsf{Ext}(F^\times, \mu_n)$. I believe that the 2-cocycle χ is the "second twist" mentioned before. Of course, since Y is a free $\mathbb{Z}$-module, this cocycle is a coboundary; still there is ambiguity up to $Hom(\tilde{Y} \otimes F^\times, \mu_n)$ in the choice of 1-chain whose coboundary is the 2-cocycle.

As the "abelian model" for metaplectic Langlands parameters, let $\mathsf{Rep}(\tilde{F}^\times, \bar{\mathbb{Q}}_\ell)$ denote the tensor category of $\bar{\mathbb{Q}}_\ell$-representations of $\tilde{F}^\times$. This tensor category is naturally graded by $Hom(\mu_n, \bar{\mathbb{Q}}_\ell^\times)$, which is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ once we choose a faithful character ϵ as before. Let $\mathsf{Rep}_m(\tilde{F}^\times, \bar{\mathbb{Q}}_\ell)$ be the m^{th} graded piece (those representations which restrict to ϵ^m on $\tilde{F}^\times$). Every continuous homomorphism $\sigma : F^\times \to \bar{\mathbb{Q}}_\ell^\times$ pulls back to given an object σ_0 of $\mathsf{Rep}_0(\tilde{F}^\times, \bar{\mathbb{Q}}_\ell)$.

For each $y \in \tilde{Y}$, the isomorphism s gives a homomorphism $y \circ s^{-1} : Q(y) \cdot \tilde{F}^\times \to \tilde{E}$, whence a pullback functor

$$p(y) = (y \circ s^{-1})^* : \mathsf{Rep}(\tilde{E}, \bar{\mathbb{Q}}_\ell) \to \mathsf{Rep}(\tilde{F}^\times, \bar{\mathbb{Q}}_\ell).$$

This, I believe, gives a parameterization of the ϵ-genuine characters $Hom(\tilde{E}, \bar{\mathbb{Q}}_\ell^\times)$ by "twisted tensor functors" from $\mathsf{Rep}(T^\vee, \bar{\mathbb{Q}}_\ell)$ to $\mathsf{Rep}(\tilde{F}^\times, \bar{\mathbb{Q}}_\ell)$, as I described in the previous letter.

For the following reason that I am interested in this cocycle perspective, even though it is not as elegant as your "naïve" approach. Consider an elliptic curve C defined over an algebraic extension $\mathbb{Q}$. Suppose that C is isogenous to each of its Galois conjugates ${}^gC$. These are called "$\mathbb{Q}$-curves" in the literature, and are studied by Gross, Ribet, Elkies, and many others. Suppose moreover that C does not have complex multiplication. Then by a theorem of Elkies (generalized to Hilbert-Blumenthal abelian varieties by Ribet), it happens that C is isogenous to an elliptic curve defined over a $(2, \ldots, 2)$-extension $K/\mathbb{Q}$ – a compositum of quadratic extensions of $\mathbb{Q}$.

Fixing this extension $K/\mathbb{Q}$, and a system of isogenies $\sigma(g) : C \to {}^gC$ for all $g \in Gal(K/\mathbb{Q})$. Let $T_\ell C$ denote the Tate module and $V_\ell = T_\ell C \otimes_{\mathbb{Z}_\ell} \bar{\mathbb{Q}}_\ell$ the resulting $\bar{\mathbb{Q}}_\ell$-sheaf on K_{et}. Then $V_\ell \cong {}^gV_\ell$ for each $g \in Gal(K/\mathbb{Q})$, via the isogenies $\sigma(g)$.

There is a central extension given by local Hilbert symbols:

$$1 \to \mu_2 \to \widetilde{\mathbb{A}^\times/\mathbb{Q}^\times} \to \mathbb{A}^\times/\mathbb{Q}^\times \to 1.$$

Pulling back via the isomorphism $W_{\mathbb{Q}}^{ab} \to \mathbb{A}^\times/\mathbb{Q}^\times$ of global class field theory yields a central extension

$$1 \to \mu_2 \to \tilde{W}_{\mathbb{Q}} \to W_{\mathbb{Q}} \to 1.$$

This central extension splits canonically over the compositum of all quadratic extensions of $\mathbb{Q}$.

Question: Does the $\mathbb{Q}$-curve C (the curve endowed with the system of isogenies) determine a representation of $\tilde{W}_{\mathbb{Q}}$ on V_ℓ?

Related question: To the data of C and the system of isogenies, can one associate a modular form of half-integral weight in a natural way (without any further choices)?

By modularity of $\mathbb{Q}$-curves, the curve C occurs as a $\bar{\mathbb{Q}}$-simple factor of $J_1(N)$ for some N; thus there is an associated weight-two newform on $\Gamma_1(N)$. By the classical Shimura correspondence, there corresponds a modular form of weight 3/2. But this process involves many choices. Can one instead pass directly from a $\mathbb{Q}$-curve to a "metaplectic Langlands parameter", to a half-integral weight modular form?

Sincerely,

Marty Weissman

Dept. of Mathematics
UC Santa Cruz

June 2, 2012

Dear Professor Deligne

I am thinking more again about metaplectic groups again. I thought that before I began a new paper, I'd send a note regarding some new ideas. Your previous comments have been most helpful to me, and perhaps this new construction is more natural than any I have written previously.

Choose a local or global field k; when k is global, write $\mathbb{A}$ for its adele ring. Write J_k for $k^\times$ if k is local, and for $\mathbb{A}^\times$ if k is global. Write C_k for $k^\times$ if k is local, and for $\mathbb{A}^\times / k^\times$ when k is global. Fix a separable algebraic closure k^{sep} of k, and let $\Gamma = Gal(k^{sep}/k)$ be the absolute Galois group.

Fix a connected reductive group $\mathbf{G}$ over k, and a central extension

$$\mathbf{K}_2 \to \mathbf{G}' \to \mathbf{G}$$

as in your paper with Brylinski. Fix an integer $n \geq 1$ such that $\#\mu_n(k) = n$. Let Ω be a ring, and

$$\epsilon : \mu_n(k) \to \Omega^\times$$

be an injective group homomorphism.

When k is local, $k \not\cong \mathbb{C}$, and $u, v \in k^\times$, define

$$(u, v)_\epsilon = \epsilon(Hilb_n(u, v))$$

to be the result of the n^{th}-order Hilbert symbol followed by ϵ. When $k = \mathbb{C}$, define $(u, v)_\epsilon = 1$. When k is global, and $u, v \in \mathbb{A}^\times$, define $(u, v)_\epsilon$ to be the product of the local Hilbert symbols followed by ϵ. Thus when k is local or global, we get a bimultipliative function

$$(\bullet, \bullet)_\epsilon : J_k \times J_k \to \mu_n(\Omega).$$

From this, I hope to construct an extension of group schemes over Ω (viewing Γ as a constant group scheme over Ω):

$$\tilde{\mathbf{G}}^\vee \to {}^L\tilde{\mathbf{G}} \to \Gamma.$$

Consider the following two twists – one is familiar from my previous work, and one is a better way of thinking of my "second twist", I hope. Hereafter let us suppose $\mathbf{G}$ is **split** over k. I hope this assumption can be removed someday soon!

First twist

Let $\mathbf{T}$ be a k-split maximal torus of $\mathbf{G}$, X its character lattice and Y its cocharacter lattice. Let Φ and $\Phi^\vee$ be the resulting roots and coroots. Let $Q : Y \to \mathbb{Z}$ be the quadratic form associated to the central extension $\mathbf{G}'$, and

$$\tilde{Y} = \{y \in Y : B_Q(y, y') \in n\mathbb{Z} \text{ for all } y' \in Y\},$$

$$\tilde{X} = \{x \in X \otimes_{\mathbb{Z}} \mathbb{Q} : \langle x, y \rangle \in \mathbb{Z} \text{ for all } y \in \tilde{Y}\}.$$

For each $\phi \in \Phi$, define

$$n_\phi = \frac{n}{GCD(n, Q(\phi^\vee))}.$$

Let $\tilde{\phi}^\vee = n_\phi \phi^\vee$ and let $\tilde{\phi} = n_\phi^{-1}\phi$. Then $\tilde{\phi}^\vee \in \tilde{Y}$ and $\tilde{\phi} \in \tilde{X}$, and these form a root system

$$(\tilde{Y}, \tilde{\Phi}^\vee, \tilde{X}, \tilde{\Phi}),$$

Let $\tilde{Y}_{sc}$ be the subgroup of $\tilde{Y}$ generated by $\tilde{\Phi}^\vee$.

Let $\left(\tilde{\mathbf{G}}^\vee, \tilde{\mathbf{B}}^\vee, \tilde{\mathbf{T}}^\vee, \{X_{\tilde{\phi}^\vee}\}\right)$ be a pinned connected reductive group over Ω with root datum $(\tilde{Y}, \tilde{\Phi}^\vee, \tilde{X}, \tilde{\Phi})$. Let $\tilde{\mathbf{Z}}^\vee = Spec(\Omega[\tilde{Y}/\tilde{Y}_{sc}])$ be the center of $\tilde{\mathbf{G}}^\vee$.

The quadratic form $Q \bmod n : Y \to \mathbb{Z}/n\mathbb{Z}$ restricts to $\tilde{Y}$, where it factors through $\tilde{Y}/\tilde{Y}_{sc}$:

$$Q \bmod n : \tilde{Y}/\tilde{Y}_{sc} \to m\mathbb{Z}/n\mathbb{Z},$$

where $m = n$ if n is odd and $m = n/2$ if n is even.

Define a function

$$z : J_k/n \times J_k/n \to \tilde{\mathbf{Z}}^\vee(\Omega) = Hom(\tilde{Y}/\tilde{Y}_{sc}, \Omega^\times)$$

by

$$[z(u, v)](y) = (u, v)_\epsilon^{Q(y) \bmod n}$$

for all $u, v \in J_k$ and $y \in \tilde{Y}/\tilde{Y}_{sc}$. Note $z(u, v) = z(v, u)$.

This function z is a two-cocycle valued in the center of $\tilde{\mathbf{G}}^\vee$, giving an extension of group schemes over Ω:

$$\tilde{\mathbf{Z}}^\vee \to {}^q\tilde{\mathbf{Z}}^\vee \to J_k/n.$$

When k is local, $J_k = C_k = k^\times$. When k is global, we have $(u, v)_\epsilon = 1$ for all $u, v \in k^\times$. This gives a canonical splitting of the above extension over $k^\times/(k^\times \cap J_k^n)$, and we obtain a central extension

$$\tilde{\mathbf{Z}}^\vee \to {}^1\tilde{\mathbf{Z}}^\vee \to C_k/n.$$

This extension ${}^1\tilde{\mathbf{Z}}^\vee$ is what we call the *first twist*. Pushing it forward gives an extension by $\tilde{\mathbf{G}}^\vee$.

$$\tilde{\mathbf{G}}^\vee \to {}^1\tilde{\mathbf{G}}^\vee \to C_k/n.$$

Second twist

Associated to the central extension $\mathbf{G}'$ of $\mathbf{G}$ by $\mathbf{K}_2$, there is a central extension

$$k^\times \to E \to Y,$$

Pulling back to $\tilde{Y} \subset Y$, and pushing forward to $k^\times/n$, we obtain a *commutative* central extension

$$k^\times/n \to \tilde{E} \to \tilde{Y}.$$

Commutativity follows from (3.11.1) of your paper with Brylinski; if n is odd, then $-1 = 1$ in $k^\times/n$; if n is even, then the bilinear form $B(y, y')$ is even for $y, y' \in \tilde{Y}$.

Also associated to the central extension $\mathbf{G}'$ of $\mathbf{G}$ by $\mathbf{K}_2$, by Theorem 6.2 of loc. cit., is a map $f : E_{sc} \to E$ (which is called ϕ in loc. cit.) making the following diagram commute:

$$\begin{tikzcd}
k^\times \arrow[r] \arrow[d, "="] & E_{sc} \arrow[r] \arrow[d, "f"] & Y_{sc} \arrow[d, "\iota"] \\
k^\times \arrow[r] & E \arrow[r] & Y
\end{tikzcd},$$

where Y_{sc} is the subgroup of Y generated by $\Phi^\vee$, and E_{sc} is the canonical central extension discussed in loc. cit.. Push this diagram forward using $k^\times \to k^\times/n$, and pull it back to $\tilde{Y}$ and $\tilde{Y}_{sc}$ (the subgroup generated by $\tilde{\Phi}^\vee$), to obtain

$$\begin{tikzcd}
k^\times/n \arrow[r] \arrow[d, "="] & \tilde{E}_{sc} \arrow[r] \arrow[d, "f"] & \tilde{Y}_{sc} \arrow[d, "\iota"] \arrow[dl, dashed, "s"] \\
k^\times/n \arrow[r] & \tilde{E} \arrow[r] & \tilde{Y}
\end{tikzcd}$$

Claim: The top row $k^\times/n \to \tilde{E}_{sc} \to \tilde{Y}_{sc}$ splits canonically.

We delay the proof of this claim to the end of this letter, focusing on the construction for now. Using the splitting (giving the arrow s in the above diagram), we get an extension of abelian groups

$$k^\times/n \to \tilde{E}/s(\tilde{Y}_{sc}) \to \tilde{Y}/\tilde{Y}_{sc}. \tag{1}$$

Define a map $k^\times/n \to Hom(C_k/n, \Omega^\times)$ by

$$u \mapsto (v \mapsto (u, v)_\epsilon).$$

Then, recalling that $\tilde{\mathbf{Z}}^\vee = Spec(\Omega[\tilde{Y}/\tilde{Y}_{sc}])$, the extension (1) gives another extension of group schemes over Ω:

$$\tilde{\mathbf{Z}}^\vee \to {}^2\tilde{\mathbf{Z}}^\vee \to C_k/n.$$

Now we have constructed two extensions ${}^1\tilde{\mathbf{Z}}^\vee$ and ${}^2\tilde{\mathbf{Z}}^\vee$ of C_k/n by $\tilde{\mathbf{Z}}^\vee$. Define ${}^L\tilde{\mathbf{Z}}^\vee$ as the Baer sum

$$ {}^L\tilde{\mathbf{Z}}^\vee = {}^1\tilde{\mathbf{Z}}^\vee \times_{\tilde{\mathbf{Z}}^\vee} {}^2\tilde{\mathbf{Z}}^\vee, $$

giving an extension

$$ \tilde{\mathbf{Z}}^\vee \to {}^L\tilde{\mathbf{Z}}^\vee \to C_k/n. $$

Push this forward, using $\tilde{\mathbf{Z}}^\vee \to \tilde{\mathbf{G}}^\vee$ to obtain a contestant for an L-group:

$$ \tilde{\mathbf{G}}^\vee \to {}^L\tilde{\mathbf{G}}^\vee \to C_k/n. $$

I believe that this agrees with my previous construction, but avoids the annoyances with cocycles everywhere. Since your constructions (of the quadratic form Q, the extension E, the function f) are functorial, I guess the Galois action can be traced through. But I cannot say much right now about non-split groups, though perhaps the tame case (e.g. groups split over an unramified extension, and covers with n coprime to the residue characteristic) can be handled without much trouble. I may try to work out some examples with nonsplit tori and nontame symbols to gather supporting evidence.

Thank you as always for any advice you can provide.

Sincerely,

Marty Weissman

Proof of claim: Consider any coroot $\tilde{\phi}^\vee = n_\phi \phi^\vee \in \tilde{\Phi}^\vee$. For any e_ϕ, $e_{-\phi}$, and n_ϕ in Tits trijection, we get an element $[e_\phi] \in E_{sc}$ projecting to $\phi^\vee \in \mathbb{Z}\Phi^\vee$. This gives an element

$$ \left([e_\phi]^{n_\phi} \in \tilde{E}_{sc} \right) \mapsto \left(\tilde{\phi}^\vee \in \tilde{Y}_{sc} \right). $$

Replacing e_ϕ by ae_ϕ for $a \in k^\times$, we have (by 11.1.9 of loc. cit.):

$$ [ae_\phi]^{n_\phi} = a^{-n_\phi Q(\phi^\vee)} \cdot [e_\phi]^{n_\phi} = [e_\phi]^{n_\phi}, $$

since $n_\phi Q(\phi^\vee) \in n\mathbb{Z}$ (by definition of n_ϕ). We have found a canonical element $[\tilde{e}_\phi] = [e_\phi]^{n_\phi} \in \tilde{E}_{sc}$ projecting onto $\phi^\vee$.

To achieve a splitting of the extension $k^\times/n \to \tilde{E}_{sc} \to \tilde{Y}_{sc}$, we must go a bit further. We utilize the long-root subsystem of $(\tilde{Y}, \tilde{\Phi}^\vee, \tilde{X}, \tilde{\Phi})$ in what follows.

Let $\tilde{\Phi}_{long}$ be the set of long roots in $\tilde{\Phi}$; if $\tilde{\Phi}$ is irreducible, the meaning is clear, considering all roots to be long if they all have the same length; now just take the union over the irreducible summands of $\tilde{\Phi}$). Let $\tilde{\Phi}^\vee_{short}$ be the set of short coroots, so if $\tilde{\alpha} \in \tilde{\Phi}_{long}$ then $\tilde{\alpha}^\vee \in \Phi^\vee_{short}$. Thus $\tilde{\Phi}^\vee_{short}$ generates $\tilde{Y}_{sc}$. Let W_{long} be the Weyl group of the long root subsystem, generated by reflections in long roots.

From Lemma 11.5 of loc. cit. (applied to the long root subsystem) we find that $\tilde{Y}_{sc}$ admits a presentation with generators $\tilde{\Phi}^\vee_{short}$ and with relations

$$s_{\tilde{\alpha}}(\tilde{\beta})^\vee = \tilde{\beta}^\vee - \tilde{\alpha}(\tilde{\beta}^\vee)\tilde{\alpha}^\vee,$$

for every $\tilde{\alpha}, \tilde{\beta} \in \tilde{\Phi}_{long}$. Choose $\tilde{\alpha}, \tilde{\beta} \in \tilde{\Phi}_{long}$, and let $\tilde{\gamma} = s_{\tilde{\alpha}}(\tilde{\beta})$. To give a splitting of $\tilde{E}_{sc}$ over $\tilde{Y}_{sc}$, it now suffices to prove the relation

$$[\tilde{e}_\gamma] = [\tilde{e}_\beta] \cdot [\tilde{e}_\alpha]^{-\tilde{\alpha}(\tilde{\beta}^\vee)}.$$

Since $n_\beta = n_\gamma$ as they are in the same W-orbit and Q is W-invariant,

$$[\tilde{e}_\gamma] = [e_\gamma]^{n_\gamma} = [int(n_\alpha)e_\beta]^{n_\beta},$$

Since $n_\alpha \tilde{\alpha}(\tilde{\beta}^\vee) = \alpha(\tilde{\beta}^\vee) = n_\beta \cdot \alpha(\beta^\vee)$,

$$[\tilde{e}_\beta] = [e_\beta]^{n_\beta}, \quad [\tilde{e}_\alpha]^{-\tilde{\alpha}(\tilde{\beta}^\vee)} = [e_\alpha]^{-n_\beta \alpha(\beta^\vee)},$$

Thus it suffices to prove that

$$[int(n_\alpha)e_\beta]^{n_\beta} = [e_\beta]^{n_\beta} \cdot [e_\alpha]^{-n_\beta \alpha(\beta^\vee)}.$$

From 11.6.1 of loc. cit., it now suffices to prove that

$$(-1)^{\epsilon(-\alpha(\beta^\vee)) \cdot Q(\alpha^\vee) \cdot n_\beta} = 1 \text{ in } k^\times/n$$

where $\epsilon(N) = N(N+1)/2$.

If n is odd, $-1 = 1$ in $k^\times/n$, so there is nothing to check, so assume n is even. If n_β is even, the result holds, so we may assume n_β is odd which implies $Q(\beta^\vee)$ is even. If $Q(\alpha^\vee)$ is even, the result holds, so we may assume $Q(\alpha^\vee)$ is odd and n_α is even. If $\alpha(\beta^\vee) = 0$, the result holds, so we may assume $\alpha(\beta^\vee) \neq 0$.

The remaining case is when $\alpha^\vee$ and $\beta^\vee$ span a root system of type B_2, with $\beta^\vee$ long and $\alpha^\vee$ short. But in this case, $\tilde{\beta}^\vee$ must be short and $\tilde{\alpha}^\vee$ must be long. This contradicts the assumption that both $\tilde{\alpha}$ and $\tilde{\beta}$ were long.

July 31, 2012

Dear Weissman,

This is a late answer to your letter of June 2.

I find the story much cleaner than before. I expect you care only about Ω of char. 0 (= a $\mathbb{Q}$-algebra). Otherwise, I think you should assume that n is invertible in Ω (injectivity of ε does not imply it). If you don't assume $\mathrm{Spec}(\Omega)$ connected, should you not assume that ε is locally an injective morphism?

I understand better the definition of the first twist when writing

$$\widetilde{Y} = \left\{ y \in Y \colon \frac{1}{n} B(y, y') \in \mathbb{Z} \text{ for all } y' \in Y \right\},$$
$$n_\phi = \text{denominator } \frac{1}{n} Q(\phi^\vee)$$

and telling that $\frac{1}{n} Q \bmod \mathbb{Z}$ is an homomorphism from Y/Y_{sc} to $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$, trivial for n odd, while for n even, if f is twice this homomorphism, with values in $\mathbb{Z}/2\mathbb{Z}$, the cocycle is $(u,v)_2^{f(y)}$.

For the second twist, after (1), you don't just use that you have a map $k^*/n \to \mathrm{Hom}(C_k/n, \Omega^*)$, but you use that it is a perfect Pontrjagin duality (hence my request that n be invertible in Ω).

Best,

P. Deligne

August 1, 2012

Dear Weissman,

I am trying to understand what you do for a split torus over k local. Take a central extension of $Y \otimes \mathbb{G}_m$ by K_2, and n, $\varepsilon\colon\ \mu_n(k) \xrightarrow{\sim} \mu_n(\Omega)$ as you do. We can pull it back to $\widetilde{Y} \otimes \mathbb{G}_m$, and get

$$
\begin{array}{ccccc}
\mu_n & \longrightarrow & E & \longrightarrow & Y \otimes k^* \\
\Vert & & \uparrow & & \uparrow \\
\mu_n & \longrightarrow & \widetilde{E} & \longrightarrow & \widetilde{Y} \otimes k^*,
\end{array}
\tag{1}
$$

where the image of $\widetilde{E}$ in E is the center of E. From $\widetilde{E}$, we get by pushing by ε a (commutative) extension

$$\Omega^* \longrightarrow E_1 \longrightarrow \widetilde{Y} \otimes k^*. \tag{2}$$

Genuine (rel. ε) irreducible representations of E give rise to splittings of this extension. This extension is "equivalent" to another one

$$\widetilde{Y}^\vee \otimes \Omega^* \longrightarrow E_2 \longrightarrow k^* : \tag{3}$$

take a basis of $\widetilde{Y}$ to see (2) as $\dim(\widetilde{Y})$ extensions of k^* by Ω^*, and similarly with (3) and check independence of basis, or more intrinsically, take $\widetilde{Y}^\vee \otimes (2)$ and a pull back by $\mathbb{Z} \to \widetilde{Y}^\vee \otimes \widetilde{Y}$. Splittings of (2) correspond one to one to splittings of (3).

Do I understand correctly that, in the split torus case, what you do is to give a description of the central extension (3), using only the data I use to describe an extension of $Y \otimes \mathbb{G}_m$ by K_2 (and n, and Hilbert symbol) ?

Best,

P. Deligne

Dept. of Mathematics
UC Santa Cruz

August 20, 2012

Dear Professor Deligne

I think it is an interesting exercise to relate the canonical extension that you described (as E_2 in the letter from August 1) to the extension I described with two twists. Indeed, they are naturally isomorphic. Here are the details.

Let $\mathbf{T}$ be a split algebraic torus over a local field k. Suppose $\mu_n = \mu_n(k)$ has n elements. Let Ω be an integral domain , with n invertible in Ω, and $\epsilon : \mu_n \to \Omega^\times$ an injective homomorphism. Let $X = \mathrm{Hom}(\mathbf{T}, \mathbf{G}_m)$ and $Y = \mathrm{Hom}(\mathbf{G}_m, \mathbf{T})$. We write ι for the canonical map $\iota : \mathbb{Z} \hookrightarrow X \otimes Y$.

Consider a central extension of $\mathbf{T}$ by $\mathbf{K}_2$, in the sense of your paper with Brylinski (hereafter cited as [BD]):

$$\mathbf{K}_2 \to \mathbf{T}' \to \mathbf{T}.$$

Let $Q : Y \to \mathbb{Z}$ be the associated quadratic form, and B_Q the bilinear form associated to Q. Let us also assume that

$$B_Q(y_1, y_2) \in n\mathbb{Z}, \text{ for all } y_1, y_2 \in Y.$$

This avoids the extra step of pulling back to what we called $\tilde{Y}$ in previous correspondence.

Let $q = Q \bmod n : Y \to \mathbb{Z}/n\mathbb{Z}$. Since $q(y_1 + y_2) = q(y_1) + q(y_2)$, we may view q as an element of $X_{/n}$.

Let $T = Y \otimes k^\times = \mathbf{T}(k)$. The central extension of $\mathbf{T}$ by $\mathbf{K}_2$ yields a central extension

$$\mu_n \to \tilde{T} \to Y \otimes k^\times.$$

You have suggested a natural way to construct an L-group. Tensor with X to obtain

$$X \otimes \mu_n \to X \otimes \tilde{T} \to X \otimes Y \otimes k^\times.$$

Pull back via $\iota : \mathbb{Z} \to X \otimes Y$ to obtain an extension I'll call E_{can}.

$$X \otimes \mu_n \to E_{can} \to k^\times.$$

Pushing forward via $\epsilon : \mu_n \to \Omega^\times$ yields a contestant L-group (over Ω) of $\tilde{T}$.

Now, I wish to show that the extension E_{can} is naturally isomorphic to the Baer sum $E_1 + E_2$, where E_1 and E_2 are central extensions described below.

The description of E_1 is straightforward; let E_1 be the trivial $(X \otimes \mu_n)$-torsor on $k^\times$, endowed with the multiplicative structure from the cocycle:

$$c_1(u,v) = q \otimes (u,v)_n, \text{ for all } u, v \in k^\times.$$

For $u \in k^\times$, write $s_1(u)$ for its lift in the trivial torsor E_1.

The description of E_2 is not as easy. The extension $\mathbf{K}_2 \to \mathbf{T}' \to \mathbf{T}$ yields, by taking $k((t))$-points, pulling back via $Y \to \mathbf{T}(k((t)))$, $y \mapsto y(t)$, and pushing forward via the tame symbol (mod n) $\partial : \mathbf{K}_2(k((t))) \to k^\times_{/n}$, an extension

$$k^\times_{/n} \to D \to Y.$$

(In [BD, 3.10.4], D would be called $\mathcal{E}$, except that we consider $k^\times_{/n}$ instead of $k^\times$.) Tensor with $k^\times$ (and note flatness of the $\mathbb{Z}$-module Y) to obtain an extension

$$k^\times_{/n} \otimes k^\times \to D \otimes k^\times \to Y \otimes k^\times.$$

Push forward via the Hilbert symbol to obtain an extension I'll call $\hat{T}$:

$$\mu_n \to \hat{T} \to Y \otimes k^\times.$$

Tensor with X:

$$X \otimes \mu_n \to X \otimes \hat{T} \to X \otimes Y \otimes k^\times.$$

Pull back via $\iota : \mathbb{Z} \to X \otimes Y$, to obtain:

$$X \otimes \mu_n \to E_2 \to k^\times.$$

The extensions E_1 and E_2 restate the two twists I discussed in a previous letter (just push forward E_1 and E_2 via $\epsilon : X \otimes \mu_n \hookrightarrow X \otimes \Omega^\times = \mathbf{T}^\vee(\Omega)$).

I think it is important to note that E_1 and E_2 are defined directly from the invariants Q and $k^\times \to D \to Y$, defined in [BD].

Claim: There is a natural isomorphism from E_{can} to the Baer sum $E_1 + E_2$.

Proof: It suffices, by [BD, §3], to assume that $\mathbf{T}'$ is a trivial $\mathbf{K}_2$-torsor over $\mathbf{T}$, with multiplicative structure given by the cocycle image of $C \in X \otimes X = Hom(Y \otimes Y, \mathbb{Z})$. Note $C(y,y) = Q(y)$ for all $y \in Y$.

For each $y \in Y$ and $u \in k^\times$, the trivialization of the $\mathbf{K}_2$-torsor $\mathbf{T}'$ gives an element $\tilde{y}(u) \in \tilde{T}$ lifting $y(u) \in T$. We write the abelian group structure on $\tilde{T}$ additively, since it will be most convenient later. From [BD], we deduce two identities:

1. $\tilde{y}_1(u) + \tilde{y}_2(u) = (u,u)_n^{C(y_1,y_2)} + \widetilde{y_1 + y_2}(u)$,
2. $\tilde{y}(u_1) + \tilde{y}(u_2) = (u_1,u_2)_n^{Q(y)} + \tilde{y}(u_1 \cdot u_2)$.

Similarly the trivialization of the $\mathbf{K}_2$-torsor $\mathbf{T}$ trivializes the $k^\times_{/n}$-torsor D over Y; for each $y \in Y$, write $\hat{y}$ for the resulting lift in D. The abelian group structure on D satisfies:

$$\hat{y}_1 + \hat{y}_2 = (-1)^{C(y_1, y_2)} + \widehat{y_1 + y_2}.$$

Of course, the extension $k^\times_{/n} \to D \to Y$ splits – but the splitting is not canonical.

Now we use an (ordered) basis $(y_1, \ldots, y_r)$ of Y to trivialize the torsors E_{can} and E_2 (noting that the torsor E_1 is trivial by construction, without any choices). Let $(x_1, \ldots, x_r)$ be the $\mathbb{Z}$-basis of X dual to $(y_1, \ldots, y_r)$.

Recall that E_{can} fits into a pullback diagram, with injective vertical arrows:

$$\begin{tikzcd} X \otimes \mu_n \arrow[r] \arrow[d, "="] & E_{can} \arrow[r] \arrow[d] & k^\times \arrow[d, "\mathbb{Z} \hookrightarrow X \otimes Y"] \\ X \otimes \mu_n \arrow[r] & X \otimes \tilde{T} \arrow[r] & X \otimes Y \otimes k^\times. \end{tikzcd}$$

In this way, we view E_{can} as a subgroup of $X \otimes \tilde{T}$. For any element $u \in k^\times$, a lift to E_{can} is given by

$$s_{can}(u) = \sum_i x_i \otimes \tilde{y}_i(u).$$

With this trivialization, the multiplicative structure on the $(X \otimes \mu_n)$-torsor E_{can} is given by the cocycle

$$c_{can}(u, v) = \sum_i x_i \otimes (u, v)_n^{Q(y_i)} = q \otimes (u, v)_n \in X \otimes \mu_n.$$

Since c_{can} equals c_1, this gives an isomorphism from E_{can} to E_1. But this isomorphism depends on the choice of basis $(y_1, \ldots, y_n)$.

Something similar is possible with the extension E_2. Begin with the extension

$$k^\times_{/n} \to D \to Y,$$

with the lifts $y \mapsto \hat{y} \in D$. Tensor with $k^\times$ to obtain the extension

$$k^\times_{/n} \otimes k^\times \to D \otimes k^\times \to Y \otimes k^\times.$$

For each $y \in Y$ and $u \in k^\times$, we have a lift $\hat{y} \otimes u \in D$ of $y \otimes u \in Y \otimes k^\times$. Push forward via the Hilbert symbol to get

$$\mu_n \to \hat{T} \to Y \otimes k^\times.$$

For $\hat{y} \otimes u \in D \otimes k^\times$, write $\hat{y}(u)$ for its image in $\hat{T}$. Now tensor with X and pullback to obtain

$$\begin{tikzcd} X \otimes \mu_n \arrow[r] \arrow[d, "="] & E_2 \arrow[r] \arrow[d] & k^\times \arrow[d, "\mathbb{Z} \hookrightarrow X \otimes Y"] \\ X \otimes \mu_n \arrow[r] & X \otimes \hat{T} \arrow[r] & X \otimes Y \otimes k^\times. \end{tikzcd}$$

For any element $u \in k^\times$, a lift to E_2 is given by

$$s_2(u) = \sum_i x_i \otimes \hat{y}_i(u).$$

With this trivialization, the multiplicative structure on the $(X \otimes \mu_n)$-torsor E_2 is given by the zero cocycle

$$c_2(u, v) = 0.$$

In other words, the section s_2 splits the extension:

$$\begin{tikzcd} X \otimes \mu \arrow[rr] & & E_2 \arrow[r] & k^\times \arrow[l, bend right, "s_2"'] \end{tikzcd}$$

Now, since $c_{can}(u, v) = c_1(u, v)$ and $c_2(u, v) = 0$, there is a unique isomorphism of extensions of $k^\times$ by $X \otimes \mu_n$

$$f : E_{can} \to E_1 + E_2,$$

such that $f(s_{can}(u))$ is the image of $(s_1(u), s_2(u))$ in the Baer sum. We write this out as

$$f\left(\sum_i x_i \otimes \tilde{y}_i(u)\right) = \left[s_1(u), \sum_i x_i \otimes \hat{y}_i(u)\right].$$

The main claim now follows if we can show that this isomorphism f does not depend on the choice of basis. So let $(y'_1, \ldots, y'_r)$ be another basis of Y, and $(x'_1, \ldots, x'_r)$ the dual basis of X. This defines another isomorphism $f' : E_{can} \to E_1 + E_2$, satisfying

$$f'\left(\sum_i x'_i \otimes \tilde{y}'_i(u)\right) = \left[s_1(u), \sum_i x'_i \otimes \hat{y}'_i(u)\right].$$

Let $\alpha = (\alpha_i^k)$ be the change of basis matrix, $\beta = (\beta_j^\ell)$ its adjoint, i.e.,

$$y_i = \sum_k \alpha_i^k y'_k, \quad x_j = \sum_\ell \beta_j^\ell x'_\ell.$$

Since we have chosen bases of X and Y dual to each other,

$$\sum_k \alpha_k^m \beta_k^n = \sum_j \alpha_m^j \beta_n^j = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases}$$

We can now use the change of basis matrix and identities in $\tilde{T}$ and $\hat{T}$ to express $\tilde{y}_i(u)$ and $\hat{y}_i(u)$ in terms of the $\tilde{y}'_j(u)$ and $\hat{y}'_j(u)$.

An explicit computation yields

$$\hat{y}_i = (-1)^{W_i} + \sum_k \alpha_i^k \hat{y}'_k, \text{ where the exponent is}$$

$$W_i = \sum_k \binom{\alpha_i^k}{2} Q(y'_k) + \sum_{1\le m<n\le r} \alpha_i^m \alpha_i^n C(y'_m, y'_n).$$

Since $\hat{y}_i(u)$ is the image of $\hat{y}_i \otimes u$ under the Hilbert symbol, we find that

$$\hat{y}_i(u) = (u,-1)_n^{W_i} + \sum_k \alpha_i^k \hat{y}'_k(u) \in \hat{T}.$$

Hence

$$\begin{aligned}
\sum_i x_i \otimes \hat{y}_i(u) &= \sum_i \sum_k x_i \otimes \alpha_i^k \hat{y}'_k(u) + \sum_i x_i \otimes (u,-1)_n^{W_i} \\
&= \sum_i \sum_{k,\ell} \alpha_i^k \beta_i^\ell (x'_\ell \otimes \hat{y}'_k(u)) + \sum_i x_i \otimes (u,-1)_n^{W_i} \\
&= \sum_{k,\ell} \left(\sum_i \alpha_i^k \beta_i^\ell \right) (x'_\ell \otimes \hat{y}'_k(u)) + \sum_i x_i \otimes (u,-1)_n^{W_i} \\
&= \sum_\ell x'_\ell \otimes \hat{y}'_\ell(u) + \sum_i x_i \otimes (u,-1)_n^{W_i}.
\end{aligned}$$

A similar computation yields (with the same W_i as before)

$$\tilde{y}_i(u) = (u,u)_n^{W_i} + \sum_k \alpha_i^k \tilde{y}'_k(u) \in \tilde{T}.$$

From this it follows that

$$\sum_i x_i \otimes \tilde{y}_i(u) = \sum_\ell x'_\ell \otimes \tilde{y}'_\ell(u) + \sum_i x_i \otimes (u,u)_n^{W_i}.$$

Therefore, recalling that f' is a homomorphism of extensions of $k^\times$ by $X \otimes \mu$,

$$\begin{aligned}
f'\left(\sum_i x_i \otimes \tilde{y}_i(u)\right) &= f'\left(\sum_\ell x'_\ell \otimes \tilde{y}'_\ell(u) + \sum_i x_i \otimes (u,u)_n^{W_i}\right) \\
&= \sum_i x_i \otimes (u,u)_n^{W_i} + \left[s_1(u), \sum_\ell x'_\ell \otimes \hat{y}'_\ell(u)\right] \\
&= \sum_i x_i \otimes (u,u)_n^{W_i} + \left[s_1(u), \sum_i x_i \otimes \hat{y}_i(u) - \sum_i x_i \otimes (u,-1)_n^{W_i}\right] \\
&= \left[s_1(u), \sum_i x_i \otimes \hat{y}_i(u)\right] \quad (\text{ since } (u,u)_n = (u,-1)_n) \\
&= f\left(\sum_i x_i \otimes \tilde{y}_i(u)\right).
\end{aligned}$$

Hence the isomorphisms f and f' are equal. We have identified an isomorphism from your canonical extension E_{can} to the Baer sum $E_1 + E_2$, that is independent of basis.

$\square$

I would be interested if there is a more direct (basis-free) way of defining the isomorphism $E_{can} \to E_1 + E_2$, to avoid the computations above. Also, the exponents W_i are interesting to me – do they have a natural interpretation? For now, I am happy that the double-twist construction of mine agrees with the much simpler construction you've mentioned for split tori.

Sincerely,

Marty Weissman